WHEN IS THE FOURIER TRANSFORM OF AN ELEMENTARY FUNCTION ELEMENTARY?

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ABSTRACT

Let F be a local field, ψ a nontrivial unitary additive character of F, and V a finite dimensional vector space over F. Let us say that a complex function on V is elementary if it has the form

$$g(x) = C\psi(Q(x))\prod_{j=1}^{k}\chi_j(P_j(x)), x \in V,$$

where $C \in \mathbb{C}$, Q a rational function (the phase function), P_j are polynomials, and χ_j multiplicative characters of F. For generic χ_j, this function canonically extends to a distribution on V (if char(F)=0).

Occasionally, the Fourier transform of an elementary function is also an elementary function (the basic example is the Gaussian integral: $k = 0$, Q is a nondegenerate quadratic form). It is interesting to determine when exactly this happens. This question is the main subject of our study.

In the first part of this paper, we show that for $F = \mathbb{R}$ or $\mathbb{C}$, if the Fourier transform of an elementary function $g \neq 0$ with phase function $-Q$ such that $\det d^2 Q \neq 0$ is another elementary function g^* with phase function Q^*, then Q^* is the Legendre transform of Q (the "semiclassical condition"). We study properties and examples of phase functions satisfying this condition, and give a classification of phase functions such that both Q and Q^* are of the form $f(x)/t$, where f is a homogeneous cubic polynomial and t is an additional variable (this is one of the simplest possible situations). Unexpectedly, the proof uses Zak's classification theorem for Severi varieties.

In the second part of the paper, we give a necessary and sufficient condition for an elementary function to have an elementary Fourier transform (in an appropriate "weak" sense) and explicit formulas for such Fourier transforms in the case when Q and P_j are monomials, over any local field F. We also describe a generalization of these results to the case of monomials of norms of finite extensions of F. Finally, we generalize some of the above results (including Fourier integration formulas) to the case when $F = \mathbb{C}$ and Q comes from a prehomogeneous vector space.

1. INTRODUCTION

1.1. **Motivations.** Let F be a local field, $\psi : F \to \mathbb{C}^*$ a nontrivial unitary additive character, V a finite dimensional vector space over F, and Q a nondegenerate quadratic form on V. It is well known that the Fourier transform of the function $\psi(-Q)$ has the form

$$\widehat{\psi(-Q)} = \epsilon(Q)\psi(Q^{-1}) \tag{1.1}$$

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where Q^{-1} is the inverse quadratic form on the dual space V^* (i.e. $dQ \circ dQ^{-1} = Id$). As was shown in [7], Proposition 3, there exists an analog of (1.1) for some homogeneous rational functions on V of homogeneity degree 2 which are not quadratic polynomials. More precisely, let E be a cyclic cubic extension of F, and $\mathcal{E} : F^* \to \mathbb{C}^*$ a nontrivial cubic character which is trivial on the image of the norm map $\mathrm{Nm} : E^* \to F^*$. Let $\phi_\mathcal{E}$ be the distribution on the vector space $F \oplus E$ such that

$$\phi_\mathcal{E}(t, x) = \mathcal{E}(t)|t|^{-1}\psi(\mathrm{Nm}(x)/t)$$

Then we have

$$\widehat{\phi_\mathcal{E}} = \epsilon\phi_\mathcal{E}, \tag{1.2}$$

where ϵ is ± 1. The proof of (1.2) given in [7] is based on the analysis of the smallest special representation of the group $D_4(F)$ and uses global arguments (such as the existence of Eisenstein series). We were interested to see whether (1.2) could be proved by local methods and whether there exist interesting generalizations of (1.2). More precisely, let us say that a distribution g on a vector space V is "elementary" if it has the form $C\psi(Q)\chi_1(P_1)....\chi_k(P_k)$, for some rational function Q (called the phase function), polynomials $P_1, ..., P_k$ on V, and multiplicative characters $\chi_1, ..., \chi_k$ on F. We say that a distribution g is "special" if both g and its Fourier transform $\hat{g}$ are "elementary". We were interested to find which distributions are "special".

In this paper we present a number of "special" distributions. Moreover, we show that the description of such distributions is almost independent of the local field F, and the Fourier transform $\hat{g}$, up to a factor, is described algebraically in terms of g. Therefore this paper provides supporting evidence for the conjecture of the existence of an algebro-geometric integration theory proposed in [8].

1.2. **Statement of the problem.** Let us formulate our main question more precisely. Keeping the notation of the previous section, set

$$G^{P_1,...,P_k,Q}_{\chi_1,...,\chi_k}(x, \psi) = \psi(Q(x)) \prod_{j=1}^{k} \chi_j(P_j(x)).$$

The function $G^{P_1,...,P_k,Q}_{\chi_1,...,\chi_k}$ is not always defined on the whole space V. However, if F has characteristic zero (and conjecturedly, also in characteristic p), this function defines a distribution on V which meromorphically depends on the characters χ_j (this follows from resolution of singularities, or from the theory of D-modules in the archimedian case). In particular, for generic values of χ_j this function canonically extends to a distribution on V.

Question: For which $Q, Q^*, P_i, P_j^*, \chi_i, \chi_j^*$ does one have

$$\widehat{G^{P_1,...,P_k,-Q}_{\chi_1,...,\chi_k}} = C_\psi G^{P_1^*,...,P_l^*,Q^*}_{\chi_1^*,...,\chi_l^*}, \forall\psi, \tag{1.3}$$

as distributions on V^*?

1.3. **The results of the paper.** The main results of the paper are as follows.

In Section 2, using the formal stationary phase method, we give a necessary condition for (1.3), when Q has nonzero Hessian. This condition says that Q^* is the Legendre transform of Q. We refer to this condition as "the semiclassical condition", commemorating the fact that identity (1.3) can be regarded as a "quantum mechanical" formula, from which this condition is deduced by using the "semiclassical" (i.e. stationary phase) approximation.

In Section 3, we discuss properties and examples of phase functions satisfying the semiclassical condition. The most interesting examples we know come from prehomogeneous vector spaces. We classify phase functions such that both Q and Q^* are of the form $f(x)/t$, where f is a homogeneous cubic polynomial and t an additional variable (Section 3). The classification says that in this case f is a relative invariant of a regular prehomogeneous vector space of degree 3 (there are seven cases). The proof of this classification theorem is based on Zak's classification theorem for Severi varieties.

In Section 4, we consider elementary functions in which the polynomials P_j and the phase function Q are monomial, and find a necessary and sufficient condition for the Fourier transform of such a function to be a function of the same type (in the "weak" sense). This condition is an identity with Γ-functions. We generalize this result to the case when Q, P_j are monomials of norms of finite extensions of F, and, more generally, when they are monomials of relative invariants of prehomogeneous vector spaces over F.

In Section 5, we show that in the archimedian case, the condition of Section 4 for the existence of an integral identity can be reformulated in combinatorial terms (more precisely, in terms of so called *exact covering systems*). We write down the explicit integral identities in the case when these combinatorial conditions are satisfied. We give the simplest nontrivial examples of integral identities, including the case of prehomogeneous vector spaces.

In Section 6, we generalize the results of Section 5 to nonarchimedian fields F. Using the known formula for the Gamma function of a cyclic field extension, we obtain integral formulas of type (1.3). In particular, we give a new proof of formula (1.2).

These results have natural analogues in the case when F is a finite field which will be described in a separate paper by D.K. and A.P. The role of distributions in this case is played by perverse sheaves and the Fourier transform is replaced by its geometric analogue defined by Deligne. Applying the trace of the Frobenius to identities with perverse sheaves, one obtains nontrivial elementary identities with exponential sums.

2. The semiclassical condition

2.1. **Formulation of the semiclassical condition.** Recall the definition of the Legendre transform (see e.g. [1]). Let V be a finite dimensional real vector space, $v_0 \in V$, and Q be a smooth function on a neighborhood of v_0 such that $\det Q''(v_0) \neq 0$. Let $Q'(v_0) = p_0 \in V^*$ (where Q', Q'' are the first and second differentials of Q). Then the Legendre transform of Q is the smooth function $L(Q)$ defined in a neighborhood of p_0 by $L(Q)(p) = pv_p - Q(v_p)$, where v_p is the unique critical point of $pv - Q(v)$ in a neighborhood of v_0.

This definition generalizes tautologically to the case when V is a vector space over any field, and Q is a regular function on the formal neighborhood of $v_0 \in V$.

It is obvious that if Q is an algebraic function then so is $L(Q)$.

Recall from Section 1.2 the definition of the function $G^{P_1,\ldots,P_k,Q}_{\chi_1,\ldots,\chi_k}(x,\psi)$. For convenience we will always assume that the pole divisor of Q is contained in the divisor $P_1\ldots P_k = 0$ (this does not cause a loss of generality).

Using the stationary phase method, we will prove the following theorem:

Theorem 2.1. *Let F be $\mathbb{R}$ or $\mathbb{C}$. Suppose that (1.3) is satisfied, and Q has nonzero Hessian. Then*

(i) The rational map of algebraic varieties $\underline{V} \to \underline{V}^*$ given by $x \to Q'(x)$ is a birational isomorphism.

(ii) Q^* is the Legendre transform of Q.

We will call this necessary condition of (1.3) **the semiclassical condition**.

Remark. We expect that a similar result holds over nonarchimedian local fields.

Theorem 2.1 is proved in Section 2.3. In the next section, we explain the formal stationary phase method, which is necessary for the proof.

2.2. **The formal stationary phase method.** The idea of the classical stationary phase method can be summarized as follows.

Let V be a real finite dimensional vector space with a volume form. Let ϕ be a function defined in an open set B around 0 in V which has a nondegenerate critical point at 0. Let f be a smooth real-valued function whose support is a compact subset of B. Consider the integral

$$I(\hbar) = \int f(x) e^{-i\phi(x)/\hbar} dx, \hbar > 0.$$

Theorem 2.2. *(see [2] and references therein) The function $I(\hbar)$ has the following asymptotic expansion as $\hbar \to 0$:*

$$I(\hbar) \sim C\hbar^{dim(V)/2}|det(\phi''(0))|^{-1/2}e^{-i\phi(0)/\hbar}(f(0) + \sum_{j=1}^{\infty} R_j(f,\phi)(i\hbar)^j),$$

where $R_j(f,\phi) = \frac{\hat{R}_j(f,\phi)(0)}{det(\phi''(0))^{N_j}}$, $N_j \in \mathbb{Z}_+$, and $\hat{R}_j$ are differential polynomials with rational coefficients.

Remark. The functions R_j are complicated, but there is an algorithm of computing them which can be expressed in terms of Feynman diagrams.

Now let p be a variable taking values in V^*. If p is small enough, the function $\phi(v) - pv$ has a unique critical point v_p near zero, which is nondegenerate. Therefore, we have

$$\int f(x) e^{i(px - \phi(x))/\hbar} dx \sim C\hbar^{dim(V)/2}|det(\phi''_p(0))|^{-1/2}e^{-i\phi_p(0)/\hbar}(f_p(0) + \sum_{j=1}^{\infty} R_j(f_p,\phi_p)(i\hbar)^j),$$

where $\phi_p(v) = \phi(v + v_p) - p(v + v_p), f_p(v) = f(v + v_p)$.

Note that $\phi_p(0) = -L(\phi)(p)$, where L is the Legendre transform.

Now we will generalize this to the formal setting. Let V be a finite dimensional vector space over a field F of characteristic zero. For any regular functions f, ϕ on a formal neighborhood of zero in V, such that $\phi(0) = 0, \phi'(0) = 0, det(\phi''(0)) \neq 0$, define the regular function $J_{f,\phi}(h,p)$ on the formal neighborhood of zero in $V^*((h))$ by

$$J_{f,\phi}(h,p) = \left(\frac{det(\phi''_p(0))}{det(\phi''(0))}\right)^{-1/2} e^{-L(\phi)(p)/h}(f_p(0) + \sum_{j=1}^{\infty} R_j(f_p,\phi_p)h^j)$$

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(this is proportional to the right hand side of the stationary phase formula, with $h = i\hbar$). We no longer claim that this series gives the asymptotic expansion of the integral $\int f(x)e^{(\phi(x)-px)/h}dx$, because this integral is not defined. However, we can still claim that the series J satisfies the same differential equations as the integral would satisfy if it existed. More precisely, we have the following Lemma, which will be used to prove Theorem 2.1.

Let D be a differential operator on V with polynomial coefficients over $F((h))$, and let $\hat{D}$ be the operator on V^* obtained from D by the Fourier automorphism $\frac{\partial}{\partial v} \to h^{-1}v$, $p \to -h\frac{\partial}{\partial p}$.

Define the differential polynomial $E_D(f, \phi)$ by $D(fe^{\phi/h}) = E_D(f, \phi)e^{\phi/h}$.

Lemma 2.3. *One has*

$$\hat{D}J_{f,\phi}(h,p) = J_{E_D(f,\phi),\phi}(h,p).$$

Proof. The statement is obvious from the stationary phase formula if $F = \mathbb{R}$ (by integration by parts), and f, ϕ are expansions of smooth functions such that f has compact support inside of the domain of ϕ. Since the statement is purely algebraic, it holds in general (because an arbitrary jet can be the jet of a function with compact support). $\blacksquare$

2.3. **Proof of Theorem 2.1.** We will consider the case $F = \mathbb{R}$; the case of $\mathbb{C}$ is similar.

Let Q, P_j, χ_j, $j = 1, .., k$ be as in Section 1.1. Let $Z \subset V_{\mathbb{C}}$ be the locus of zeros of P_i. The function $g(x) := G^{P_1,...,P_k,Q}_{\chi_1,...,\chi_k}(x, \psi)$ is smooth and nonvanishing on $V_{\mathbb{C}} \setminus Z$ and hence generates a 1-dimensional local system on $V_{\mathbb{C}} \setminus Z$. Let us extend this local system to an irreducible D-module on $V_{\mathbb{C}}$ and call this extension M_g. (Note that all our D-modules are algebraic D-modules).

Consider the D-module $M(r)$ generated by the distribution $g_r := (P_1...P_k)^r g$ for sufficiently large r. Since $M(r)$ is holonomic, it has finite length, and so for sufficiently large r the D-module $M(r)$ is independent of r. Let us denote this D-module by M_∞.

It is easy to see that there is an exact sequence of D-modules

$$0 \to K \to M_\infty \to M_g \to 0,$$

where K is supported on the divisor $P_1...P_k = 0$. This implies that M_g is isomorphic to D_V/I_r (for large enough r), where D_V is the algebra of differential operators on V, and I_r is the left ideal of differential operators which annihilate g_r formally (i.e. outside of Z).

Proposition 2.4. *For generic ψ, the rank of the Fourier transform of M_g is at least the degree d of the map $Q' : \underline{V} \to \underline{V^*}$.*

Proof. Let us change the ground field from $\mathbb{C}$ to $K = \overline{\mathbb{C}((\hbar))}$, and set formally $\psi(x) = e^{ix/\hbar}$. It is enough to prove the claim for this particular ψ.

For this purpose, it is enough to produce, for a generic $p_0 \in V^*_{\mathbb{C}}$, a collection of d linearly independent solutions of the differential equations $\hat{D}\phi = 0$, $D \in I_r$, in the formal neighborhood of p_0.

To produce such solutions, we will use the formal stationary phase method. We will pick p_0 generically. Then the equation $Q'(x) = p_0$ has exactly d distinct

solutions $x_1, ..., x_d$, and Q'' is nondegenerate at all these points. Define power series
$\phi_i(x) = Q(x + x_i) - Q(x_i) - p_0 x$, $f_i(x) = \prod_j \chi_j(P_j)(P_1...P_k)^r(x + x_i)$. Define

$$\eta_j(\hbar, p) = J_{f_j, \phi_j}(i\hbar, p).$$

It follows from Lemma 2.3 that these series are indeed solutions of the equations
$\hat{D}(p + p_0)\phi(p) = 0$, $D \in I_r$.

It remains to show that the solutions η_i are linearly independent. To do this, consider the power series $L(\phi_i)(p)$, and look at their second degree terms, which are equal to $Q''(x_i)^{-1}(p, p)$.

Lemma 2.5. *For generic p_0, the forms $Q''(x_i)^{-1}$ are distinct.*

Proof. The forms $Q''(x_i)^{-1}$ are all the values at $p = p_0$ of the multivalued algebraic function $((Q')^{-1})'(p)$, which is the derivative of the function $(Q')^{-1}(p)$ that has exactly d branches by the definition.

But we claim that the derivative of any algebraic vector-function has at least (and hence exactly) as many branches as the function itself. Indeed, it is enough to check it for scalar functions $f(z)$ of one variable. But in the one variable case, one always has $f \in \mathbb{C}(z, f')$, since any monic algebraic equation of degree > 1 satisfied by f over $\mathbb{C}(z, f')$ can be differentiated to get a monic equation of lower degree. The lemma is proved. $\square$

Now let us make a change of variables $\hbar \to t^2\hbar$, $p \to tp$, and let t tend to 0. Then we have $\eta_j \to e^{iQ''(x_j)^{-1}(p,p)/\hbar}$, which are linearly independent functions since $Q''(x_j)^{-1}$ are distinct. This implies that η_i are linearly independent. The Proposition is proved. $\square$

Let $G_1 = G_{\chi_1, ..., \chi_k}^{P_1, ..., P_k, -Q}$, $G_2 = G_{\chi_1^*, ..., \chi_l^*}^{P_1^*, ..., P_l^*, Q^*}$.

Corollary 2.6. *Suppose that (1.3) holds. Then for generic ψ the Fourier transform of the D-module M_{G_1} is isomorphic to M_{G_2}.*

Proof. The relation $\hat{G}_1 = C_\psi G_2$ implies that the Fourier transform of the D-module generated by G_1 is the D-module generated by G_2. Both of these D-modules are holonomic and have only one component of the Jordan-Holder series which has nonzero rank, namely, M_{G_1} and M_{G_2}. But by Proposition 2.4, the rank of the Fourier transform of M_{G_1} is positive. This implies the statement. $\square$

Now let us prove the theorem. We start with statement (i). By Proposition 2.4, for generic ψ the rank of the Fourier transform M_{G_1} is at least d. On the other hand, by Corollary 2.6 this Fourier transform is M_{G_2}, so its rank is 1. So $d = 1$, as desired.

Statement (ii) follows from the fact that G_2 satisfies (outside of Z) the differential equations $\hat{D}f = 0$, where $D \in I_r$, and hence (since the rank of the Fourier transform of M_{G_1} is 1), the formal expansion of G_2 must coincide with the formal series constructed using the stationary phase method. This implies that $Q^* = L(Q)$, since it is clear that if $J_{f,\phi}(h, p) = e^{-\xi(p)/h}(1 + O(p, h))$ then $\xi = L(\phi)$ (this follows from theorem 2.2). The Theorem is proved.

Remark. It is known that Theorem 2.2 has a nonarchimedian analogue, which is even simpler than this theorem itself: in this case, the asymptotic expansion of the integral contains only the leading term and no higher terms. We expect that this result can be used to prove Theorem 2.1 in the nonarchimedian case.

2.4. **Integral identities in the weak sense and the semiclassical condition.**
Let F be archimedian, and V be a finite dimensional vector space over F. Let P be a polynomial on V and R a polynomial on V^*. Let N be a positive integer, and $\mathcal{S}_N^{P,R}(V)$ be the space of Schwartz functions on V of the form $|P|^{2N}f$ (where f is a Schwartz function), whose Fourier transform has the form $|R|^{2N}g$ (where g is a Schwartz function). It is easy to construct examples of elements of this space: for instance, one can take the function $|R|^{2N}(\partial)|P|^{2N'}(x)f$, where $N' >> N$, and f is any Schwarz function.

As we mentioned before, the function $G_{\chi_1,\ldots,\chi_k}^{P_1,\ldots,P_k,Q}$ defines a distribution only for generic values of the characters χ_j. However, for any characters χ_j this function defines a linear functional on the space $\mathcal{S}_N^{P_1\ldots P_k,1}(V)$ for large enough N.

Definition. We will say that the integral identity (1.3) holds **in the weak sense** if it holds on the space $\mathcal{S}_N^{P_1^*\ldots P_l^*,P_1\ldots P_k}(V^*)$ for large enough N.

Theorem 2.7. *Theorem 2.1 remains valid if (1.3) holds only in the weak sense.*

The proof is analogous to the proof of Theorem 2.1.

3. RATIONAL FUNCTIONS SATISFYING THE SEMICLASSICAL CONDITION.

In this section we would like to study systematically the question: which rational functions satisfy the semiclassical condition?

3.1. **Properties of functions satisfying the semiclassical condition.** Let V be a finite dimensional vector space over an algebraically closed field F. Denote by $SC(V)$ the set of rational functions Q on V^* satisfying the semiclassical condition, i.e. such that $Q' : V \to V^*$ is a birational isomorphism.

One can characterize elements of $SC(V)$ using the notion of Legendre transform, as follows.

Proposition 3.1. *A function $Q \in F(V)$ belongs to $SC(V)$ if and only if the Legendre transform of Q is rational.*

Proof. The "only if" part is obvious. To prove the "if" part, let $L(Q) = G$ and let us differentiate the equation $xQ'(x) - Q(x) = G(Q'(x))$. We get $xQ''(x) = G'(Q'(x))Q''(x)$. Since Q'' is generically nondegenerate, we get $x = G'(Q'(x))$. Thus, G' is the inverse to Q'. $\qquad\square$

Nondegenerate quadratic forms are the simplest examples of elements of $SC(V)$. In the following sections we will construct other examples of elements of $SC(V)$.

3.2. **The projective semiclassical condition.**
Definition. A homogeneous rational function f on V is said to satisfy *the projective semiclassical condition* if the map $x \to f'(x)$ defines a birational isomorphism $\mathbb{P}V \to \mathbb{P}V^*$.

Denote the set of functions satisfying the projective semiclassical condition by $PSC(V)$.

The relationship between $SC(V)$ and $PSC(V)$, which motivates the introduction of $PSC(V)$, is given by the following easy lemma.

Lemma 3.2. *Let $g : V \to W$ be a homogeneous of degree d rational map of finite dimensional vector spaces, which defines a birational isomorphism $\bar{g} : \mathbb{P}V \to \mathbb{P}W$. Then g itself is a birational isomorphism if and only if $d = \pm 1$. In particular, an*

element $Q \in PSC(V)$ belongs to $SC(V)$ if and only if its homogeneity degree is 0 or 2.

Proof. It is enough to prove the first statement; the second statement is a special case of the first one for $g = Q'$.

If. The condition of the lemma implies that for a generic vector v one has $v = tR(g(v))$, where R is a rational function, and t is a factor to be determined. Thus we have $t^d g(R(g(v))) = g(v)$, which allows to determine t rationally since $d = \pm 1$.

Only if. This part is clear, since a homogeneous birational isomorphism between vector spaces has to have homogeneity degree 1 or -1. $\square$

Corollary 3.3. *A homogeneous function $f \in F(V)$ belongs to $PSC(V)$ if and only if the map $f'/f : V \to V^*$ is a birational isomorphism.*

Proof. The "if" part is clear. The "only if" part follows when one applies Lemma 3.2 to $g = f'/f$. $\square$

Corollary 3.4. *(i) Any nonzero (in F) integer power of a function $f \in PSC(V)$ belongs to $PSC(V)$.*

(ii) Let V, W be finite dimensional F-vector spaces, and $f \in PSC(V)$, $g \in PSC(W)$. Then the exterior tensor product $(f \otimes g)(v, w) = f(v)g(w)$ on $V \oplus W$ belongs to $PSC(V \oplus W)$.

(iii) Any function of the form $f_1^{n_1}...f_k^{n_k}$, where $f_i \in PSC(V_i)$, and $n_i \in \mathbb{Z}$ are nonzero in F, belongs to $PSC(\oplus_i V_i)$; it belongs to $SC(\oplus_i V_i)$ iff $\sum n_i d_i = 0$ or 2, where d_i are homogeneity degrees of f_i.

Proof. Statements (i) and (ii) are obvious from Corollary 3.3. Statement (iii) follows from (i),(ii), and Lemma 3.2. $\square$

Part (iii) of Corollary 3.4 allows one to obtain functions satisfying the semiclassical condition from functions satisfying the projective semiclassical condition. The simplest example of functions so obtained are monomial functions $x_1^{n_1}...x_k^{n_k}$, where $\sum n_i = 0$ or 2.

So we will now study the projective semiclassical consition more systematically.

3.3. **The multiplicative Legendre transform.** To think about elements of $PSC(V)$, it is useful to introduce the notion of the multiplicative Legendre transform.

Let f be a homogeneous function, and $det((f'/f)')$ is not identically zero. In this case we can define a function f_* by $f_*(f'/f(x)) = 1/f(x)$ (as the usual Legendre transform, it can be defined in an analytic as well as a formal setting). If f is homogeneous of degree d then so is f_*.

Definition. We will call f_* the multiplicative Legendre transform of f.

Remark. Our terminology is motivated by the fact that f_* is the multiplicative Legendre transform of f if and only if $L(\ln f) = d + \ln f_*$ (over $\mathbb{C}$).

It is obvious that the operation $f \to f_*$ commutes with exterior tensor product. Also, $(f^n)_* = n^{-nd} f_*^n$, where d is the degree of f (if $n \neq 0$ in F).

Example. $(\prod x_i^{n_i})_* = \prod n_i^{-n_i} \prod x_i^{n_i}$.

Proposition 3.5. *(i) $f_*'/f_* \circ f'/f = Id$.*
(ii) $f_{**} = f$.

Proof. The first identity is obtained by differentiating the definition of f_*. The second one is obtained by applying f_{**} to both sides of (i). $\quad\square$

One can characterize elements of $PSC(V)$ using the notion of the multiplicative Legendre transform, as follows.

Proposition 3.6. *A homogeneous rational function f belongs to $PSC(V)$ if and only if f_* is rational.*

Proof. The "only if" part follows from Corollary 3.3. The "if" part follows from Proposition 3.5 (i). $\quad\square$

Remark. We see from the above that elements of $PSC(V)$ are multiplicative analogs of elements of $SC(V)$.

Example. Let us point out an easy method of creating new functions satisfying the projective semiclassical condition out of ones already known. It is straightforward to compute that if f_* is the multiplicative Legendre transform of f on V then the multiplicative Legendre transform of the function $\mathbb{F}(x, y) = f'(x)y + f(x)$ on V^2 is $\mathbb{F}_*(x_*, y_*) = (d-1)^{1-d}(f'_*(y_*)x_* - f_*(x_*))^{d-1}f_*(y_*)^{2-d}$. This formula is valid also for $d = 1$ if we agree that $0^0 = 1$.

Remark. Note that if $\mathbb{F}$ is a polynomial and $d \geq 3$ then $\mathbb{F}_*$ is not a polynomial. Construction of polynomial elements $f \in PSC(V)$ such that f_* is also a polynomial is more tricky, and the only examples we know are described in the next section.

3.4. **Construction of elements of $PSC(V)$ from prehomogeneous vector spaces.** Recall [11, 10] that a prehomogeneous vector space over F is a triple (G, V, χ), where G is an algebraic group over F, V an algebraic representation of G, and χ a nontrivial algebraic character of G such that

(i) V has a Zariski dense G-orbit, and

(ii) there exists a nonzero polynomial f on V such that $f(gv) = \chi(g)f(v)$, $g \in G$ (it is obvious that if condition (i) holds, f is unique up to a scalar).

Here we will assume that the group G is reductive.

Remark. Prehomogeneous vector spaces were introduced in the 60-s by Sato (see [11]). Prehomogeneous vector spaces over $\mathbb{C}$ with reductive G and irreducible representation V have been classified, see [10].

Let $G_0 = Ker(\chi)$. Then G_0 is a codimension 1 subgroup of G, and the function f is invariant under G_0. The following proposition characterizes the ring of all G_0-invariants.

Proposition 3.7. [10] *The ring of G_0-invariants of V is $F[f]$.*

Corollary 3.8. *G_0 has a dense orbit in $\mathbb{P}V$.*

Proof. (of Proposition 3.7)

Let h be any homogeneous polynomial which is an eigenfunction for G and invariant under G_0. Since $G/G_0 = \mathbb{G}_m$, there exist nonzero integers k, l such that $h^k f^l$ is G-invariant. Since the G-action has a dense orbit, this implies that $h^k f^l$ is a constant, i.e. h, f are powers of the same polynomial g, which is invariant under G_0 and is an eigenfunction of G. By the definition of f, this polynomial has to be f (we can't have $g^k = f$ for $k > 1$ since then there exist elements of G which are not in G_0 but preserve f). The proposition is proved. $\quad\square$

The function f (which is uniquely determined up to scaling) is called the relative invariant of (G, V, χ).

From now till the end of the subsection we will assume that the characteristic of F is zero.

Recall [11, 10] that a regular prehomogeneous vector space is such that $det(f'')$ is not identically zero.

Proposition 3.9. *(see* [11]*) Let (G, V, χ) be a regular prehomogeneous vector space, and f its relative invariant. Then $f \in PSC(V)$, and its multiplicative Legendre transform is a polynomial.*

Proof. It is clear that (G, V^*, χ^{-1}) is a prehomogeneous vector space. Indeed, the existence of an open orbit, and the existence of a relative invariant of the same degree d follows from the fact that the representation $S^k V^*$ is completely reducible.

Let f_* be the relative invariant of (G, V^*, χ^{-1}). Consider the function $f_* \circ f'$ on V. This function is nonzero because of regularity, has degree $d(d-1)$, and and G_0-invariant. Thus, this function is proportional to f^{d-1}. So f_* is the multiplicative Legendre transform of f up to scaling. The proposition is proved. $\square$

Examples. 1. $G = GL(1)$, $V = F$, $\chi(a) = a^n$, $f(x) = x^n$.

2. $G = GL(1)^n$, $V = F^n$, the action is $(a_1, ..., a_n)(x_1, ..., x_n) = (a_1 x_1, ..., a_n x_n)$, $\chi(a) = a_1...a_n$, $f = x_1...x_n$.

3. $G = GL(n)$, $V = S^2 F^n$, where F^n is the vector representation, $\chi = det^2$, $f = det$.

4. $G = GL(n) \times GL(n)$, $V = F^n \otimes (F^n)^*$, $\chi = det \otimes det^{-1}$, $f = det$.

5. $G = GL(2n)$, $V = \Lambda^2 F^{2n}$, $\chi = det$, $f = Pf$ (the Pfaffian).

6. $G = E_6 \times GL(1)$, V is the 27-dimensional irreducible representation (with $GL(1)$ acting by scalar multiplication), χ is $z \to z^3$, f is the invariant cubic form.

7. $G = O(N) \times GL(1) \times GL(1)$, $V = F^N \oplus F$, where F^N is the vector representation of $O(N)$, the action $(g, a, b)(v, x) = (agv, bx)$, $\chi(g, a, b) = a^2 b$, $f(v, x) = Q(v)x$, where Q is the invariant quadratic form.

To conclude this subsection, we would like to raise two questions.

Question 1. Is it true that any polynomial $f \in PSC(V)$ such that f_* is also a polynomial, is a relative invariant of a prehomogeneous vector space?

Question 2. Is it true that any polynomial $f \in PSC(V)$ such that f_* is also a polynomial, is rigid? In other words, is the set of equivalence classes of such polynomials finite for each degree?

Remark. Clearly, a positive answer to question 1 implies a positive answer to question 2, but question 2 could be more tractable.

For degree ≤ 3, the answer to both questions is yes. This is proved in the next subsection.

3.5. **Classification of cubic forms with cubic multiplicative Legendre transform.** In this section the field F has characteristic zero. We prove the following theorem.

Theorem 3.10. *Let f be a cubic form on a finite dimensional vector space V such that its multiplicative Legendre transform is also a cubic form. Then f is given by one of Examples 1-7 of the previous section.*

The rest of the section is the proof of this theorem.

First of all, we may assume that f (and f_*) are irreducible (in which case $dim(V) \geq 3$). If any of them is reducible, it is a product of a linear and a quadratic

form or of three linear forms, and it is easy to see that it is given by examples 1,2, or 7.

The functions f, f_* satisfy the equations

$$f_*(f') = f^2, f(f'_*) = f_*^2, f' \circ f'_*(x) = f_*(x)x, f'_* \circ f'(x) = f(x)x. \qquad (3.1)$$

Let $X \subset V$ be the zero locus of f. Let $Z \subset X$ be the zero locus of (f, f'), i.e. the singular locus of X. Define $Z_* \subset X_* \subset V^*$ in a similar way.

Lemma 3.11. *(i)* $Z = (f')^{-1}(0)$, $Z_* = (f'_*)^{-1}(0)$.
  (ii) $X = (f')^{-1}(Z_*)$, $X_* = (f'_*)^{-1}(Z)$.

Proof. Part (i) follows from the fact that $f' = 0$ implies $f = 0$, and similarly for f_* (Euler equation). Part (ii) follows from equations (3.1). $\qquad\square$

Definition. A 3-dimensional subspace in V is said to be a Cremona subspace if it contains three noncoplanar lines (through 0) in Z and is not entirely contained in X.

Cremona subspaces in V^* are defined similarly.

Lemma 3.12. *Let L be a Cremona subspace in V. Then $f'(L)$ is a Cremona subspace in V^*, and the map $f' : L \to f'(L)$ is the Cremona map $(x, y, z) \to (yz, zx, xy)$ in some coordinates. In particular, the intersection of a Cremona subspace with Z is the union of the three lines from the definition.*

Proof. Let z_1, z_2, z_3 be noncoplanar lines in Z which are contained in L, and $p_3 = z_1 z_2$, $p_2 = z_1 z_3$, $p_1 = z_2 z_3$ be the planes spanned by pairs of lines. Consider the restriction of the map f' to p_1. This map is given by a homogeneous quadratic form of two variables with values in V^*, and it vanishes at lines z_2, z_3 by Lemma 3.11. Therefore, $f'|_{p_1}$ has the form $q(v)w$, where q is a quadratic function, and $w \in V$. The same applies to p_2, p_3. Let z_1^*, z_2^*, z_3^* be the images of p_1, p_2, p_3. Then $z_i^* \subset Z_*$ ($p_i \subset X$ because X is a cubic) and each z_i^* is a line or zero. Let L_* be a 3-subspace in V^* containing all z_i^*. Let l be a generic plane in L. Then $f'(L)$ has three intersection lines with L_* (images of intersections of l with p_1, p_2, p_3). Since f' is a quadratic map, we have $f'(l) \subset L_*$ and thus $f'(L) \subset L_*$.

Let us show that z_i^* are in fact nonzero (i.e. lines) and noncoplanar. If z_i^* lie in a 2-plane p, $f'_* \circ f'|_L$ is a map from a 3-space to a 2-plane. Therefore, L must be contained in X, as it is obvious from (3.1) that $f'_* \circ f'$ is finite (4 to 1) outside of X. This contradicts the definition of L.

This shows that L_* is a Cremona subspace (it is clear that L_* is not entirely contained in X_*), and it is obvious that if z_i, z_i^* are used as coordinate axes then f' becomes the Cremona map. The lemma is proved. $\qquad\square$

Proposition 3.13. *A generic plane p (through 0) in V is contained in a Cremona subspace.*

Proof. Let a_1, a_2, a_3 be the three lines of intersection of p with X. Let $z_i^* = f'(a_i) \in Z_*$. It is clear that z_i^* are noncoplanar, since otherwise $p \subset X$, and we assumed that p is generic. Thus the 3-space L_* spanned by z_i^* is a Cremona subspace. Let $L = f'_*(L_*)$. Then by Lemma 3.12 L is a Cremona subspace which contains a_i. $\qquad\square$

Proposition 3.14. $dim(Z) \geq \frac{2}{3} dim(V) - 1$, *and Z is not contained in any hyperplane.*

Proof. Define a map $g : Z \times Z \times Z \times F^3 \to V \oplus V$ by $G(\zeta_1, \zeta_2, \zeta_3, b_1, b_2, b_3) = (\sum \zeta_i, \sum b_i \zeta_i)$. This map is dominant, since by Proposition 3.13 two generic vectors are contained in a Cremona subspace. This implies the Proposition. $\square$

Proposition 3.15. *Z is irreducible.*

Proof. It is enough to show that the map $f'_* : X_* \to Z$ is surjective, i.e. that for any $\zeta \neq 0$, $\zeta \in Z$ there exists $x \in X_*$ such that $f'_*(x) = \zeta$. To do this, it is enough to show that any such ζ is contained in a Cremona subspace, since then ζ lies in the image of a special plane in the dual Cremona subspace in X_*.

Suppose that $\zeta_1, \zeta_2 \in Z$ are such that $f''(\zeta)(\zeta - \zeta_1, \zeta - \zeta_2) \neq 0$. Then $f(\zeta + t(\zeta - \zeta_1) + s(\zeta - \zeta_2))$ is of order ts modulo cubic and higher terms at $t, s = 0$, and thus the points $0, \zeta, \zeta_1, \zeta_2$ are not in the same plane and span a Cremona subspace.

Thus, if the proposition was false, we would have $f''(\zeta)(\zeta_1 - \zeta, \zeta_2 - \zeta) = 0$ for all choices of ζ_1, ζ_2. But since Z does not lie in any hyperplane, the possible vectors $\zeta - \zeta_1$ span V, and similarly for $\zeta - \zeta_2$. Thus, $f(\zeta) = f'(\zeta) = f''(\zeta) = 0$. This implies that $f(y + \zeta) = f(y)$ for all $y \in V$, and hence f is pulled back from $V/ < \zeta >$. This is a contradiction, since we assumed that $det(f'')$ is not identically zero. $\square$

Proposition 3.16. $dim(Z) = \frac{2}{3} dim(V) - 1$ *(in particular, d is divisible by 3), and $Z \setminus 0$ is smooth.*

Proof. Since Z is irreducible, to prove the first statement it is sufficient to show that the map g defined in the proof of Proposition 3.14 is generically finite. For this, it suffices to show that the Cremona subspace containing a generic plane is unique. Indeed, if p is a generic plane, a_1, a_2, a_3 is as in the proof of Proposition 3.13, then a_i are contained in any Cremona subspace containing p, so $f'(a_i) = z_i^*$ are contained in the image of this subspace. But z_i^* are noncoplanar, so such a subspace is unique. The first statement is proved.

Let us now prove that Z is smooth. The dimension of the generic fiber of the map $f'_* : X_* \setminus Z_* \to Z \setminus 0$ is $dim(X_*) - dim(Z) = d/3$, where $d = dim(V)$. Consider $df'_* : V^* \to V$. It is enough to show that the nullity of $df'_* = f''_*$ is $\leq d/3$ at all points of $X_* \setminus Z_*$ (then it is exactly $d/3$ everywhere, and π is a smooth fiber bundle).

Now we will need two simple lemmas.

Lemma 1. $det(f''_*) = const \cdot f^{d/3}$.

Proof. This follows from the facts that this determinant is nonzero outside of X (where f'_* is a 2-1 covering), the irreducibility of f, and the fact that $deg(det f''_*) = d$.

Lemma 2. Suppose that $A(t)$ is a polynomial family of matrices such that $det A(t)$ vanishes exactly to d-th order at $t = 0$. Then the nullity of $A(0)$ is at most d.

Proof. We can assume that the kernel of $A(0)$ is the span of the first r basis vectors, so the first r columns of A vanish at 0. Thus, $det(A) = O(t^r)$, so $r \leq d$.

Let us now apply Lemma 2 to f''_* restricted to a line transversal to X_* at a nonsingular point. Taking into account Lemma 1, we get that the nullity of f''_* at this point is $\leq d/3$, which completes the proof of the proposition. $\square$

Now we will finish the proof of the theorem. The above propositions shows that the projectivization $\mathbb{P}(Z)$ of Z is a smooth projective variety of dimension $\frac{2}{3}(dim\mathbb{P}(V) - 2)$ in $\mathbb{P}(V)$, which does not lie in a hyperplane. It has the following property: the line connecting any two points of Z is entirely contained in X and

thus never goes through a generic point of $\mathbb{P}(V)$. Varieties with these properties are called Severi varieties.

Now comes the central part of the proof, which is the use of the following classification theorem of Severi varieties, due to F.Zak.

Theorem 3.17. [13] *Let Y is a smooth, closed subvariety of the complex projective space $\mathbb{C}P^{d-1}$, which does not lie in a hyperplane. Suppose that*
(i) any line connecting two points of Y belongs to a certain hypersurface, and
(ii) $dim(Y) = \frac{2}{3}(d-3)$.
Then Y is projectively equivalent to the singularity locus of the equation $f = 0$ on $\mathbb{P}(V)$, where (V, f) is one of the following four prehomogeneous vector spaces with relative invariant:
1. V is the space of 3 by 3 symmetric matrices, $f = det$.
2. V is the space of 3 by 3 matrices, $f = det$.
3. V is the space of skew-symmetric 6 by 6 matrices, $f = Pf$.
4. V is a 27-dimensional irreducible representation of E_6, f is the invariant cubic form.

This theorem shows that in our situation, (V, f) is given by one of the examples 3-6 in the previous section.

The variety X is reconstructed from Z as the set of points on lines connecting two points of Z. The theorem is proved. $\qquad\square$

Corollary 3.18. *Let $Q(x, t) = f(x)/t$, where f is a homogeneous cubic polynomial on some finite dimensional complex vector space W, and t an additional variable. Then the following conditions are equivalent:*
(i) $Q \in SC(W \oplus \mathbb{C})$, and $Q^* = \tilde{f}(x_*)/t_*$, where $\tilde{f}$ is a cubic polynomial on W^*.
(ii) The polynomial f is as in Examples 1-7.

Proof. It is easy to check directly that $L(f(x)/t) = \tilde{f}(x_*)/t_*$ iff $\tilde{f} = -f_*$. Thus the statement follows from Theorem 3.10. $\qquad\square$

Remark 1. In examples 3-5, the map g of proposition 3.14 has a classical linear algebraic interpretation. Example 3 corresponds to simultaneous diagonalization of two quadratic forms. Example 4 corresponds to simultaneous diagonalization of two hermitian forms. Example 5 corresponds to simultaneos reduction of two skewsymmetric forms to the canonical form (sum of 2-dimensional forms).

Remark 2. Examples 2-7 correspond to the semisimple Jordan algebras of degree 3 [6]. It is possible that one can check directly (i.e. without using Theorem 3.10) that in the assumptions of Theorem 3.10, if $dim(V) > 1$ then V admits a structure of a separable (hence semisimple) Jordan algebra of degree 3 such that f is its determinant polynomial. This would allow to give another proof of Theorem 3.10 which would use Albert's theorem on classification of simple Jordan algebras, rather than Zak's classification theorem.

4. Integral identities with monomials

4.1. **Fourier transform.** In the next four sections we will recall some basic facts about analysis over local fields. The basic reference for these facts is the book [4].

Let F be a local field. We fix a nontrivial additive character ψ. For any finite dimensional vector space V over F, let $\mathcal{S}(V)$ be the space of (complex-valued) Schwartz functions on V.

For any Haar measure dx on V, one defines the Fourier transform $\mathcal{S}(V) \to \mathcal{S}(V^*)$ by

$$\hat{f}(y) = \int_F \psi(yx)f(x)dx. \tag{4.1}$$

Any Haar measure on V defines a positive inner product on $\mathcal{S}(V)$. Let us say that Haar measures dx on V and dx^* on V^* are compatible if the Fourier transform is a unitary operator with respect to this inner product. It is easy to see that this condition is symmetric, and that if it is satisfied then one has the inversion formula $\hat{\hat{f}}(x) = f(-x)$.

If V is identified with V^*, one can choose a unique Haar measure which is compatible with itself. For example, this is so if $V = F$ or F^n. From now on, we will use the notation dx for this special measure.

The measure dx on F depends on ψ. For example, in the archimedian case, if $\psi(x) = e^{i\mathrm{Re}(ax)}$ then dx is $(|a|/2\pi)^{\dim_{\mathbb{R}} F/2}$ times the Lebesgue measure. Below, we use the character $\psi(x) = e^{i\mathrm{Re}(x)}$ for $F = \mathbb{R}, \mathbb{C}$, and a character of norm 1 for the non-archimedian case. This completely determines dx.

Let $\mathcal{D}(V)$ be the space of distributions on V. If V carries a Haar measure dx, then along with the Fourier transform of Schwartz functions, one can define Fourier transform of distributions $\mathcal{D}(V^*) \to \mathcal{D}(V)$ (by duality). It will also be denoted by $f \to \hat{f}$. In the case of compatible measures, it satisfies the inversion formula.

4.2. **Multiplicative characters.** Let F^* be the multiplicative group of F, $U(F^*)$ the group of continuous unitary characters of F^*, and $X(F^*)$ the group of continuous characters of F^* into $\mathbb{C}^*$. If $F = \mathbb{R}$, then $U(F^*) = \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ and $X(F^*) = \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$. If $F = \mathbb{C}$, then $U(F^*) = \mathbb{R} \times \mathbb{Z}$, and $X(F^*) = \mathbb{C} \times \mathbb{Z}$. In the nonarchimedian case, $U(F^*)$ is $\mathbb{R}/\mathbb{Z} \times D$, where D is a discrete countable group, and $X(F^*) = \mathbb{C}/\mathbb{Z} \times D$.

Let $d_m x$ be the multiplicative Haar measure on F^*, normalized so that $d_m x/dx = 1$ at $x = 1$. Let $\nu_F(x) = (d_m x/dx)^{-1}$ be the norm of x (in the archimedian case, it equals $|x|^{\dim_{\mathbb{R}} F}$).

For $\lambda \in X(F^*)$, denote by $\mathrm{Re}\,\lambda$ the real number defined by $|\lambda(x)| = \nu_F(x)^{\mathrm{Re}\,\lambda}$. It is obvious that $X(F^*) = U(F^*) \times \mathbb{R}$, via $\chi \to (\frac{\chi}{|\chi|}, \mathrm{Re}\,\chi)$. We think of the first coordinate as the imaginary part of χ and of the second as the real part of χ.

Let us say that $\lambda \in X(F^*)$ is a singular character if $\lambda(x) = \nu_F(x)^{-1}$ (in the nonarchimedian case), $\lambda(x) = \nu_F(x)^{-1}x^{-n}$, $n \in \mathbb{Z}_{\geq 0}$ (for $F = \mathbb{R}$), and $\lambda(x) = \nu_F(x)^{-1}x^{-n}\bar{x}^{-m}$, $n, m \in \mathbb{Z}_{\geq 0}$ (for $F = \mathbb{C}$). It is well known that $\lambda(x)$ is a holomorphic family of distributions on F depending on $\lambda \in X(F^*)$, with simple poles at singular characters.

4.3. **Gamma functions.** Now define the Gamma function $\Gamma^F(\lambda)$ of a local field F to be the meromorphic function on $X(F^*)$ given by

$$\widehat{\lambda\nu_F^{-1}}(x) = \Gamma^F(\lambda)\lambda^{-1}(x), \tag{4.2}$$

(whenever both $\lambda\nu_F^{-1}$ and λ^{-1} define distributions on F).

From the inversion formula for Fourier transform one gets the functional equation

$$\Gamma^F(\lambda)\Gamma^F(\nu_F\lambda^{-1}) = \lambda(-1) \tag{4.3}$$

Let us give the expressions for the Gamma functions of $\mathbb{R}$ and $\mathbb{C}$.

Lemma 4.1. *For $s \in \mathbb{C}$, let $\lambda_{s,n}(x) = |x|^s (x/|x|)^n$, $x \in F$. Let $\Gamma_n^F(s) = \Gamma^F(\lambda_{s,n})$. Then*

$$\Gamma_n^{\mathbb{R}}(s) = (2\pi)^{-1/2} \cdot 2i^n \Gamma(s) \cos(\pi(s-n)/2), \ n \in \mathbb{Z}/2\mathbb{Z}, \qquad (4.4)$$

and

$$\Gamma_n^{\mathbb{C}}(s) = (2\pi)^{-1} 2^s i^n \Gamma(\frac{s+n}{2})\Gamma(\frac{s-n}{2}) \sin(\pi(s-n)/2), \ n \in \mathbb{Z}. \qquad (4.5)$$

This lemma is well known and is proved in a straightforward way.

One can also easily compute the Gamma function of a power of ν_F in the nonarchimedian case. If q is the order of the residue field of F then

$$\Gamma^F(\nu_F^s) = \frac{1 - q^{s-1}}{1 - q^{-s}}. \qquad (4.6)$$

From these formulas it is clear that the Gamma function has simple poles at the characters $\lambda\nu_F$, where λ is singular. It is clear from the definition that it is holomorphic everywhere else. (For an exact expression of the Gamma function in the non-archimedian case, see [4]).

4.4. **Integral representation of the additive character.** Let du be the Haar measure on $U(F^*)$ for which the Mellin transform $L^2(F^*, d_m x) \to L^2(U(F^*), du)$ is a unitary operator.

We would like to consider a distribution on $U(F^*)$ of the form

$$\phi \to \int_{U(F^*)} \Gamma^F(u)\phi(u)du.$$

Since $\Gamma^F(u)$ has a pole at the trivial character, it is necessary to choose a regularization of this integral. Our choice, here and throughout, will be the following: to avoid the pole, the contour of integration in the connected component of the identity in $X(F^*)$ should be indented in the direction of positive values of $\mathrm{Re}(u)$. The following lemma shows that this choice coincides with the choice of [4], where $\Gamma^F(u)$ is defined as the Mellin transform of ψ.

Lemma 4.2. *The distribution $\psi(x)$ on F has the following integral representation:*

$$\psi(x) = \int_{U(F^*)} \Gamma^F(u)u^{-1}(x)du,$$

Remark. This integral is divergent for any concrete value of x but is absolutely convergent on any test function from the Schwartz space.

Proof. It is easy to see from the definition of the Gamma function that the Fourier transform on the group F^* of the distribution $\psi(x)$ (i.e. the Mellin transform) is equal to Γ^F outside of 0. Therefore the statement of the lemma on test functions which vanish at 0 is obtained by applying the inversion formula for Fourier transform. It suffices now to check this identity on one test function which does not vanish at 0. In the nonarchimedian case, this is easy to do for the characteristic function of the integers, and in the archimedian case one can do it for the function $e^{-|x|^2/2}$. $\qquad\square$

4.5. **The generalized Gamma function.** For a positive integer d, define a mero-morphic function $\Gamma^F_{d,a}$ on $X(F^*)$ by the formula

$$\Gamma^F_{d,a}(\lambda) = d^{-1} \sum_{\mu:\mu^d=\lambda} \Gamma^F(\mu)\mu^{-1}(a).$$

We have $\Gamma^F_{1,a}(\lambda) = \Gamma^F(\lambda)\lambda^{-1}(a)$, so the function $\Gamma^F_{d,a}$ is a generalization of the Gamma function. We will call $\Gamma^F_{d,a}$ the generalized Gamma function.

Let us compute the generalized Gamma function in the archimedian case.

If $F = \mathbb{C}$ then

$$\Gamma^F_{d,a}(\lambda_{s,nd}) = \Gamma^F(\lambda_{s/d,n})\lambda_{-s/d,-n}(a), \tag{4.7}$$

If $F = \mathbb{R}$ and $d = 2k+1$ then

$$\Gamma^F_{d,a}(\lambda_{s,n}) = \frac{1}{d}\Gamma^F(\lambda_{s/d,n})\lambda_{-s/d,-n}(a). \tag{4.8}$$

If $F = \mathbb{R}$, $\psi(x) = e^{ix}$, $a > 0$, and $d = 2k$ then

$$\Gamma^F_{d,\pm a}(\lambda_{s,0}) = (2\pi)^{-1/2}k^{-1}e^{\pm\pi is/4k}\Gamma(s/2k)a^{-s/2k}. \tag{4.9}$$

4.6. **The distribution $G^{n_1,...,n_k}_{\lambda_1,...,\lambda_k,a}$ on F^k and its integral representation.** Let $n_1,...,n_k$ be integers, and $\lambda_1,...,\lambda_k \in X(F^*)$. Consider the function on $(F^*)^k$ defined by

$$G^{n_1,..,n_k,a}_{\lambda_1,...,\lambda_k}(x_1,...,x_k) = \psi(a\prod_{i=1}^{k} x_i^{n_i})\lambda_1(x_1)...\lambda_k(x_k). \tag{4.10}$$

(Here $a \in F^*$ is a parameter).

$G^{n_1,..,n_k,a}_{\lambda_1,...,\lambda_k}$ is a distribution on F^k which is holomorphic in λ_i if $\mathrm{Re}\,\lambda_i > -1$.

Lemma 4.3. *For λ_i with real parts > -1 one has*

$$G^{n_1,..,n_k,a}_{\lambda_1,...,\lambda_k}(x_1,...,x_k) = \int_{U(F^*)} \Gamma^F(u)u^{-1}(a)\lambda_1 u^{-n_1}(x_1)...\lambda_k u^{-n_k}(x_k)du.$$

(in the sense of distributions).

Proof. Follows directly from Lemma 4.2. $\qquad\qquad\square$

It will be convenient for us to understand the function $G^{n_1,...,n_k,a}_{\lambda_1,...,\lambda_k}$ as a distribution **in the weak sense**. Namely, for polynomials P, R define the space $\mathcal{S}^{P,R}_N(V)$ as in section 2 for the archimedian case, and as the space of Schwartz functions vanishing on the variety $P = 0$ whose Fourier transform vanishes on $R = 0$, in the non-archimedian case (so in the non-archimedian case it is independent of N). As in Section 2, for any λ_i the function $G^{n_1,...,n_k,a}_{\lambda_1,..,\lambda_k}$ defines a linear functional on the space $\mathcal{S}^{P_1...P_k}_N(F^k)$ for a large enough N. We call such a functional a distribution in the weak sense.

The following lemma is a straghtforward generalization of the previous lemma.

Lemma 4.4. *For any λ_i one has*

$$G^{n_1,..,n_k,a}_{\lambda_1,...,\lambda_k}(x_1,...,x_k) = \int_{U(F^*)} \Gamma^F(u)u^{-1}(a)\lambda_1 u^{-n_1}(x_1)...\lambda_k u^{-n_k}(x_k)du.$$

(as distributions in the weak sense).

4.7. The Fourier transform of the distribution $G^{n_1,...,n_k,a}_{\lambda_1,...,\lambda_k}$.

Lemma 4.5. *One has*

$$\widehat{G^{n_1,...,n_k,a}_{\lambda_1,...,\lambda_k}}(p_1,...,p_k) =$$

$$\int_{U(F^*)} \Gamma^F(u)u^{-1}(a)\Gamma^F(\lambda_1 u^{-n_1}\nu_F)...\Gamma^F(\lambda_k u^{-n_k}\nu_F)\lambda_1^{-1}u^{n_1}\nu_F^{-1}(p_1)...\lambda_k^{-1}u^{n_k}\nu_F^{-1}(p_k)du.$$

Proof. This follows from the previous lemma. $\qquad\square$

4.8. Identities with monomials.

Theorem 4.6. *Let $n_1,...n_k$, $m_1,...,m_k$ be nonzero integers. Let $d = gcd(n_1,...,n_k)$. Then the identity*

$$\widehat{G^{n_1,...,n_k,a}_{\lambda_1,...\lambda_k}} = CG^{m_1,...,m_k,b}_{\eta_1,...,\eta_k}, \tag{4.11}$$

between distributions on F^k in the weak sense, is satisfied if and only if

$$\eta_i\lambda_i\nu_F = \gamma^{n_i}. \tag{4.12}$$

where $\gamma \in X(F^*)$ is a character, and one of the following two conditions holds:
   1. $\sum m_i = 2$, $m_i = n_i$, and

$$\Gamma^F_{d,a}(u^d)\prod_{i=1}^{k}\Gamma^F(u^{-n_i}\lambda_i\nu_F) = C\Gamma^F_{d,b}(u^{-d}\gamma^d); \tag{4.13}$$

   2. $\sum m_i = 0$, $m_i = -n_i$, and

$$\Gamma^F_{d,a}(u^d)\prod_{i=1}^{k}\Gamma^F(u^{-n_i}\lambda_i\nu_F) = C\Gamma^F_{d,b}(u^d\gamma^{-d}). \tag{4.14}$$

Proof. Using the previous two lemmas, we get

$$\int_{U(F^*)} \Gamma^F(u)u^{-1}(a)\Gamma^F(\lambda_1 u^{-n_1}\nu_F)...\Gamma^F(\lambda_k u^{-n_k}\nu_F)\lambda_1^{-1}u^{n_1}\nu_F^{-1}(p_1)...\lambda_k^{-1}u^{n_k}\nu_F^{-1}(p_k)du =$$

$$C\int_{U(F^*)} \Gamma^F(v)v^{-1}(b)\eta_1 v^{-m_1}(p_1)...\eta_k v^{-m_k}(p_k)dv.$$

We see that this identity can hold only if the vectors $(m_1,...,m_k)$ and $(n_1,...,n_k)$ are proportional. Let $\alpha = m_i/n_i$ for all i (the proportionality coefficient). It is not difficult to see by asymptotic analysis of the above formula for $u,v = \nu_F^s$ for large s (which is essentially equivalent to the "semiclassical analysis" of Section 2) that $1 - \sum m_i = \alpha$. Therefore, $1 - \sum n_i = \alpha^{-1}$, so both α and α^{-1} are integers and hence $\alpha = \pm 1$. Thus, $m_i = \pm n_i$.

So we should consider two cases.

Case 1. $\sum m_i = 2$, $m_i = n_i$. In this case, replacing v with v^{-1}, we obtain

$$\int_{U(F^*)} \Gamma^F(u)u^{-1}(a)\Gamma^F(\lambda_1 u^{-n_1}\nu_F)...\Gamma^F(\lambda_k u^{-n_k}\nu_F)\lambda_1^{-1}u^{n_1}\nu_F^{-1}(p_1)...\lambda_k^{-1}u^{n_k}\nu_F^{-1}(p_k)du =$$

$$C\int_{U(F^*)} \Gamma^F(v^{-1})v(b)\eta_1 v^{n_1}(p_1)...\eta_k v^{n_k}(p_k)dv.$$

Let us replace u with $u^{1/d}$. This leads to summation over all roots of degree d, and therefore the Gamma functions $\Gamma^F(u)$, $\Gamma^F(v)$ are replaced by the generalized Gamma functions:

$$\int_{U(F^*)} \Gamma^F_{d,a}(u)\Gamma^F(\lambda_1 u^{-n_1/d}\nu_F)...\Gamma^F(\lambda_k u^{-n_k/d}\nu_F)\lambda_1^{-1}u^{n_1/d}\nu_F^{-1}(p_1)...\lambda_k^{-1}u^{n_k/d}\nu_F^{-1}(p_k)du =$$

$$C\int_{U(F^*)} \Gamma^F_{d,b}(v^{-1})\eta_1 v^{n_1/d}(p_1)...\eta_k v^{n_k/d}(p_k)dv.$$

It is clear that the integrals on the two sides of this equation can coincide if and only if the contour of integration in the first integral can be shifted to obtain the second integral. In particular, there must exist a character γ such that $\eta_i\lambda_i\nu_F = \gamma^{n_i}$. In this case, replacing v with $u\gamma^{-d}$ (i.e. shifting the contour of integration), we get

$$\int_{U(F^*)} \Gamma^F_{d,a}(u)\Gamma^F(\lambda_1 u^{-n_1/d}\nu_F)...\Gamma^F(\lambda_k u^{-n_k/d}\nu_F)\lambda_1^{-1}u^{n_1/d}\nu_F^{-1}(p_1)...\lambda_k^{-1}u^{n_k/d}\nu_F^{-1}(p_k)du =$$

$$C\int_{U(F^*)} \Gamma^F_{d,b}(u^{-1}\gamma^d)\lambda_1 u^{n_1/d}\nu_F^{-1}(p_1)...\lambda_k u^{n_k/d}\nu_F^{-1}(p_k)du.$$

Remark. We can shift the contour of integration without worrying about residues, since our integral identities are understood in the weak sense, while residual contributions are distributions supported on the coordinate hyperplanes, which by definition do not affect identities in the weak sense.

The latter condition is equivalent to

$$\Gamma^F_{d,a}(u)\Gamma^F(\lambda_1 u^{-n_1/d}\nu_F)...\Gamma^F(\lambda_k u^{-n_k/d}\nu_F) = C\Gamma^F_{d,b}(u^{-1}\gamma^d).$$
(4.15)

Since $\Gamma_{d,a}$, by definition, vanishes away from d-th powers, we can replace in this condition the variable u by u^d without changing the condition. This yields the equation (4.14) in the theorem.

Case 2. $\sum m_i = 0$, $m_i = -n_i$. In this case, we see similarly to case 1 that (4.11) is equivalent to the combination of the two conditions from part 2 of the theorem.

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.9. **Gamma functions of prehomogeneous vector spaces.** The construction of the previous section can in fact be further generalized to a more general setting of prehomogeneous vector spaces.

For simplicity let F have characteristic zero. Let $\underline{V}$ be a prehomogeneous vector space of dimension M over F for a reductive group $\underline{G}$. We assume that $V = V(F)$ has a dense orbit under the action of the group of points $G = G(F)$, and that the same is true about the dual prehomogeneous vector space V^*.

Let f be a relative invariant of V generating its invariant ring. Suppose it has degree D. Let V^* be the dual space and f_* be the relative invariant of V^*, which is the multiplicative Legendre transform of f. Assume for simplicity that the stabilizer G_0 of f in G acts with finitely many orbits on the varieties $f = 0$, $f_* = 0$ in V, V^*.

For any multiplicative character λ of F^*, consider the function $\lambda(f(x))$ on V. Since F has characteristic zero, it is known that this function defines a distribution on V which meromorphically depends on λ.

It is easy to see that for generic λ the Fourier transform $\widehat{\lambda\nu_F^{-\frac{M}{D}}(f)}(x)$ is proportional to $\lambda^{-1}(f_*(x))$. (For generic λ, this is the unique, up to scaling, distribution

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of the correct homogeneity degree, due to our assumption about the finiteness of the number of orbits). This allows one to define the Gamma function $\Gamma^V(\lambda)$ of V to be the meromorphic function on $X(F^*)$ given by

$$\widehat{\lambda\nu_F^{-\frac{M}{D}}(f)}(x) = \Gamma^V(\lambda)\lambda^{-1}(f_*(x)), \tag{4.16}$$

(whenever both sides define distributions on F).

From the inversion formula for Fourier transform one gets the functional equation

$$\Gamma^V(\lambda)\Gamma^V(\nu_F^{M/D}\lambda^{-1}) = \lambda(-1)^D \tag{4.17}$$

In fact, it was shown by Sato and Shintani [12] that if F is archimedian, then these results are valid without the assumption of the finiteness of the number of orbits.

4.10. **Identities with monomials on prehomogeneous vector spaces.** Let $V_1, ..., V_k$ be prehomogeneous vector spaces over F like in the previous section, of dimensions M_i, and f_i be their relative invariants, of degrees D_i. Let $V = \oplus V_i$.

By a monomial on V we will mean a rational function on V of the form $f_1(x_1)^{n_1}...f_k(x_k)^{n_k}$, where n_i are integers, and $x_i \in V_i$. Define the distribution on V by the formula

$$\mathcal{G}_{\lambda_1,...,\lambda_k}^{n_1,...,n_k,a}(x_1,...,x_k) = \psi(a\prod_{i=1}^{k} f_i(x_i)^{n_i})\lambda_1(f_1(x_1))...\lambda_k(f_k(x_k)). \tag{4.18}$$

Here $a \in F^*$ is a parameter, and $\lambda_i \in X(F^*)$. Similarly to the case of usual monomials, this is a distribution for large real parts of λ_i which meromorphically extends to generic λ_i, and a distribution in the weak sense for all λ_i.

For this class of distributions, we have the following generalization of Theorem 4.6.

Theorem 4.7. *Let $n_1,...n_k$, $m_1,...,m_k$ be nonzero integers. Let $d = gcd(n_1,...,n_k)$. Then the identity*

$$\widehat{\mathcal{G}_{\lambda_1,...\lambda_k}^{n_1,...,n_k,a}} = C\mathcal{G}_{\eta_1,...,\eta_k}^{m_1,...,m_k,b}, \tag{4.19}$$

between distributions $V^* = \oplus V_i^*$ in the weak sense is satisfied if and only if

$$\eta_i \lambda_i \nu_F^{M_i/D_i} = \gamma^{n_i}. \tag{4.20}$$

where $\gamma \in X(F^*)$ is a character, and one of the following two conditions holds:
   1. $\sum m_i D_i = 2$, $m_i = n_i$,

$$\Gamma_{d,a}^F(u^d)\prod_{i=1}^{k}\Gamma^{V_i}(u^{-n_i}\lambda_i\nu_F^{M_i/D_i}) = C\Gamma_{d,b}^F(u^{-d}\gamma^d); \tag{4.21}$$

   2. $\sum m_i D_i = 0$, $m_i = -n_i$, *and*

$$\Gamma_{d,a}^F(u^d)\prod_{i=1}^{k}\Gamma^{V_i}(u^{-n_i}\lambda_i\nu_F^{M_i/D_i}) = C\Gamma_{d,b}^F(u^d\gamma^{-d}). \tag{4.22}$$

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Remark. Here the distribution $\mathcal{G}_{\eta_1,\ldots,\eta_k}^{m_1,\ldots,m_k,b}$ is defined using the relative invariants f_i^* of V_i^*, which are multiplicative Legendre transforms of f_i.

Proof. The proof of this theorem is analogous to the proof for usual monomials. □

An important special case of this theorem describes integral identities with monomials of norms of field extensions, which includes the identity from [7] cited in Section 1.1. This special case is defined by setting $V_i = F_i$ (field extensions of F of degrees d_i), and $f_i = \mathrm{Nm} : F_i \to F$ to be the norm maps In this case, V_i is identified with V_i^* using the trace functional.

4.11. Poles of distributions $G_{\lambda_1,\ldots,\lambda_k}^{n_1,\ldots,n_k,a}$.

We would like to find out when integral identities with monomials hold not only in the weak but also in the strong sense (i.e. on all test functions). For this purpose we should first of all find out where (in terms of λ_i) the distribution $G_{\lambda_1,\ldots,\lambda_k}^{n_1,\ldots,n_k,a}$ could have poles. This is our goal in this section. For brevity we will denote this distribution just by G.

Proposition 4.8. *The divisor of poles of G is a subset of the set of points where one of the following equations is satisfied:*

(i) λ_i is a singular character for some i such that $n_i \geq 0$;

(ii) $(\lambda_j r_j^{-1})^{n_l} = (\lambda_l r_l^{-1})^{n_j}$ for some (j,l) such that $n_j > 0 > n_l$, and singular characters r_j, r_l.

(The notion of a singular character was introduced in Section 4.2).

Proof. It is clear that we can assume that all the exponents n_i are nonzero.

Suppose that a point $(\lambda_1, ..., \lambda_k) \in X(F^*)^k$ is a generic point of the divisor of poles of G. Then the leading coefficient G_{top} of the Laurent expansion of G at that point is a distribution on the coordinate cross. Its support is a closed subset S of the coordinate cross which is invariant under the scaling group $T = \{(t_1, ..., t_k) \in (F^*)^k : \prod t_i^{n_i} = 1\}$.

Let $H_I = \{x_i = 0, i \notin I\}$, $I \subset \{1, ..., k\}$. Assume that λ_i is not a singular character whenever $n_i > 0$. Then $S \subset \cup_{I:|I|=k-2} H_I$.

Let y be a generic point of S. In this case the T-orbit of y spans a certain subspace H_I, $|I| \leq k-2$. In a small neighborhood of y, the sets S and H_I coincide.

Let T_I be the group of elements of T in which $t_i = 1$, $i \in I$ (so T_I acts trivially on H_I). On one hand, for $t \in T_I$ we have $G_{top}(tx) = \prod_{i \notin I} \lambda_i(t_i) G_{top}(x)$. On the other hand, since in the neighborhood of y the distribution G_{top} is supported on H_I, there exist singular characters r_j, $j \notin I$ such that $G_{top}(tx) = \prod_{i \notin I} r_i(t_i) G_{top}(x)$. This implies that whenever $\prod_{i \notin I} t_i^{n_i} = 1$, we have $\prod_{i \notin I} (\lambda_i r_i^{-1})(t_i) = 1$, for suitable singular characters r_i. In particular, for any distinct $i, j \notin I$, one has $(\lambda_i r_i^{-1})^{n_j} = (\lambda_j r_j^{-1})^{n_i}$.

The last equation can define a component of the pole divisor only if n_j, n_l have opposite signs, since otherwise there are solutions with $\mathrm{Re}(\lambda_j) > 0$, $\mathrm{Re}(\lambda_l) > 0$, where G is holomorphic. The proposition is proved. □

4.12. Identities with monomials in the weak and strong sense.

Sometimes one can deduce from an integral identity in the weak sense that it actually holds in the strong sense (i.e. on all test functions). Let us do it in the case of archimedian fields, using the theory of D-modules.

Let F be archimedian. Let us say that a nonsingular multiplicative character λ is *strongly regular* if it generates (as a distribution) an irreducible D-module on the line. For $F = \mathbb{R}$ a character is strongly regular iff it is not of the form $x^n/|x|$ for

an integer n, or x^n for $n < 0$. For $F = \mathbb{C}$, a character is strongly regular iff it is not of the form $x^n \bar{x}^m$, where n, m are integer, and at least one of them is negative.

Theorem 4.9. *Suppose that for some k-tuples of characters $(\lambda_1, ..., \lambda_k)$, $(\eta_1, ..., \eta_k)$ identity (4.11) holds in the weak sense, and*

(i) λ_i, η_i are strongly regular whenever $n_i \geq 0$;

(ii) $(\lambda_j r_j^{-1})^{n_l} \neq (\lambda_l r_l^{-1})^{n_j}$ for any (j, l) such that $n_j > 0 > n_l$, and any singular characters r_j, r_l, and the same is true about η_i.

In this case both sides of (4.11) are well defined as distributions (on all test functions), and (4.11) is satisfied as an identity between distributions (in the "strong" sense).

Proof. According to Proposition 4.8, the distributions $G_1 = G_{\lambda_1, ..., \lambda_k}^{n_1, ..., n_k, a}$, and $G_2 = G_{\eta_1, ..., \eta_k}^{m_1, ..., m_k, b}$ are well defined. The fact that (4.11) holds in the weak sense means that $\hat{G}_1 - G_2 = \Delta$, where Δ is the sum of a distribution supported on the coordinate cross and a distribution whose Fourier transform is supported on the cross. Let M_i be the D-modules generated by G_i.

Lemma. M_1 and M_2 are irreducible.

Proof. Let us prove that M_1 is irreducible. The irreducibility of M_2 is shown in a similar way.

Let $I \subset \{1, ..., k\}$. We claim that if the numbers $n_i, i \in I$, have the same signs, then the D-module M_1 is irreducible on the formal polydisk around a generic point y of the subspace H_I (where H_I was defined in the proof of Proposition 4.8). Indeed, if all $n_i, i \in I$ are positive, this restriction is isomorphic to the exterior tensor product of $< \lambda_i >, i \in I$ with $k - |I|$ copies of $< 1 >$ (the structure sheaf of the 1-dimensional formal disk), so it is irreducible by assumption (i). If all n_i are negative, the restriction is generated by $\prod_{j \in I} \lambda_j(x_j) e^{i \operatorname{Re}(c(x) \prod_{j \in I} x_j^{n_j})}$, where c depends only of $x_j, j \notin I$, and is nonzero at y. This D-module is obviously irreducible for any λ_j.

Thus, we see that all Jordan-Holder components of M_1 whose support is contained in the cross, are supported on the union of subspaces $H_{I_{j,l}}$ for $n_j > 0 > n_l$, where $I_{j,l}$ is the complement of $\{j, l\}$. Now, using scaling arguments as in the proof of Proposition 4.8, it is easy to deduce from condition (ii) for j, l that all these components are zero. This means that M_1 is irreducible. The lemma is proved. $\square$

Now let us prove the theorem. By the irreducibility of M_1, Δ is supported on the cross (as well as its Fourier transform), since the restriction of M_1 to the set of its smooth points has to be an irreducible local system.

Lemma. The distribution Δ has support of codimension 2 or more.

Proof. If the support of Δ contains a point $(x_1, ..., x_k)$ with $x_i = 0$, $x_j \neq 0$ for $j \neq i$ then let us apply to $\hat{G}_1$ the algebra of differential operators in one variable x_i, with coefficients in polynomials of other variables. It suffices to check that the space of obtained distributions contains a nonzero distribution supported on the cross (this would contradict to the irreducibility of M_2).

For this, it suffices to check that for any distribution on the line of the form $h = \chi(t)\psi(ct^m) + \Delta$, where $m \neq 0$, χ is strongly regular if $m > 0$, and Δ is supported at zero, $\Delta \neq 0$, one can find a polynomial differential operator D (algebraically depending on c) such that Dh is nonzero but is supported at zero. This is straightforward. $\square$

Finally, since $\mathrm{Supp}(\Delta)$ has codimension 2 or more, it is not difficult to see using homogeneity arguments that $\Delta = 0$.

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark. The same method can be used for identities with norms and prehomogeneous vector spaces; however, we do not give here the details of such arguments.

5. Identities over $\mathbb{R}$ and $\mathbb{C}$.

As we saw in the previous section, the Gamma functions for $\mathbb{R}$ and $\mathbb{C}$ are given by simple explicit formulas via the Euler Gamma function. One can also compute the Gamma functions of prehomogeneous vector spaces over these fields, using the theory of Bernstein's polynomials. This allows one to express the Gamma function identities of the previous section in much more explicit (combinatorial) terms, thus giving a completely elementary criterion of the existence of integral identities. This is what we do in this section.

For the sake of brevity, we do a systematic analysis only in the case of ordinary monomials with relatively prime exponents, restricting ourselves to a number of examples in other cases.

5.1. The relation of the parameters a and b.

Lemma 5.1. *Suppose that identity (4.21) is satisfied in the weak sense. Then if $\sum m_i = 2$, one has*

$$ab = -\prod_i n_i^{-n_i D_i}, \qquad\qquad (5.1)$$

and if $\sum m_i = 0$, one has

$$ab^{-1} = \prod_i n_i^{-n_i D_i} \qquad\qquad (5.2)$$

Proof. This follows from Theorem 2.1 or from analyzing the asymptotics of the Gamma function identity for large s. $\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

5.2. Multiplication formulas for Γ-functions. We recall the classical multiplication law for Γ-function:

$$\Gamma(Nz) = N^{Nz-\frac{1}{2}}(2\pi)^{(1-N)/2}\Gamma(z)\Gamma(z+1/N)...\Gamma(z+(N-1)/N).$$
$$(5.3)$$

This implies that

$$\Gamma_{Nn}^{\mathbb{C}}(Ns) = N^{Ns-1}\Gamma_n^{\mathbb{C}}(s)\Gamma_n^{\mathbb{C}}(s+2/N)...\Gamma_n^{\mathbb{C}}(s+2(N-1)/N), \qquad (5.4)$$

and if N is odd then

$$\Gamma_{Nn}^{\mathbb{R}}(Ns) = i^{-(N-2)(N-1)/2}N^{Ns-1/2}\Gamma_n^{\mathbb{R}}(s)\Gamma_{n+1}^{\mathbb{R}}(s+1/N)...\Gamma_{n+N-1}^{\mathbb{R}}(s+(N-1)/N),$$
$$(5.5)$$

This can be uniformly written as

$$\Gamma^F(\chi^N) = C_{F,N}\chi(N^N)\Gamma^F(\chi)\Gamma^F(\chi\mathrm{Nm}^{1/N})...\Gamma^F(\chi\mathrm{Nm}^{(N-1)/N}), \qquad (5.6)$$

(N is odd for $F = \mathbb{R}$), where Nm is the algebraic norm of F as an extension of $\mathbb{R}$ ($\mathrm{Nm}(x) = x\bar{x}$ for $\mathbb{C}$ and $\mathrm{Nm}(x) = x$ for $\mathbb{R}$), and $C_{F,N} = N^{-1}$ for $\mathbb{C}$ and $i^{-(N-2)(N-1)/2}N^{-1/2}$ for $\mathbb{R}$.

5.3. Classification of identities with monomials.

Define the group $X_F = X(F^*)/ < \mathrm{Nm} >$. Let $Div(X_F)$ be the group of divisors on X_F. Now for $\chi \in X_F$ and a positive integer N set $D_{\chi,N} = (\chi) + (\chi \mathrm{Nm}^{1/N}) + ... + (\chi \mathrm{Nm}^{(N-1)/N}) \in Div(X_F)$ (here N should be odd if $F = \mathbb{R}$). Set $D_{\chi,-N} = -D_{\chi^{-1}\nu_F, N}$.

Theorem 5.2. *(i) Make the assumptions of Theorem 4.6, and suppose that the exponents n_i are relatively prime: $d = 1$. Then identity (4.11) holds in the weak sense for some C, a, b, λ_i, η_i in and only in one of the following two situations:*

1. $\sum m_i = 2$, $m_i = n_i$, and there exists $\xi, \mu_i \in X_F$ such that:

$$D_{1,1} + \sum_i D_{\mu_i,-n_i} = D_{\xi,-1}. \tag{5.7}$$

2. $\sum m_i = 0$, $m_i = -n_i$, and there exists $\xi, \mu_i \in X_F$ such that:

$$D_{1,1} + \sum_i D_{\mu_i,-n_i} = D_{\xi,1}. \tag{5.8}$$

(ii) More precisely, in any of the above two cases, one can find $\lambda_i, \gamma \in X(F^*)$, $a, b \in F^*$ (where γ is as in Theorem 4.6), so that the integral identity of Theorem 4.6 is satisfied, and $\lambda_i \nu_F = \mu_i$, $\gamma = \xi$ (resp. $\gamma = \xi^{-1}$) modulo Nm in the first (resp. second) case.

Proof. Let us sketch the proof of (i). The proof of (ii) is obtained automatically in the process of proving (i).

To prove the necessity of the conditions of (i), let us represent the elements of the group X_F in $X(F^*)$ by a fundamental domain lying in the region $\mathrm{Re}\, u >> 0$. Let us write down the identity of divisors of zeros and poles for the left and right hand sides of the two Gamma function relations of Theorem 4.6 in this fundamental domain. It is easy to check that this yields exactly the two divisorial relations above.

Let us now prove the sufficiency of (i). It is not difficult to show that if the conditions of (i) are satisfied, then one can find elements $\lambda_i, \gamma \in X(F^*)$ in the cosets $\mu_i \nu_F^{-1}, \xi^{\pm 1}$ of the free cyclic group $< \mathrm{Nm} >$, such that the divisors of both sides of the Gamma function relations of Theorem 4.6 in $X(F^*)$ (not only in X_F) coincide.

By the multiplication formula (5.6), this implies that these relations hold up to a constant and an exponential factor. These factors can be removed by an appropriate choice of the constant C, and by imposing relations (5.1),(5.2) on a and b. $\qquad\square$

5.4. Identities with monomials and exact covering systems.

Let $(p_1, ..., p_n)$ be positive integers whose sum equals p. According to a classical definition in elementary number theory ([5], problem F14) an exact covering system of type $(p_1, ..., p_n)$ is a representation of the group $\mathbb{Z}/p\mathbb{Z}$ as a disjoint union of cosets of groups $\mathbb{Z}/p_i\mathbb{Z}$, $i = 1, ..., n$. It is clear that for the existence of an exact covering system of type $(p_1, ..., p_n)$, it is necessary that all p_i are divisors of p. But this is not sufficient: for example, it is clear that for a covering system, p/p_i is never relatively prime to p/p_j. In fact, there is no known simple necessary and sufficient condition for the existence of an exact covering system, and various questions about exact covering systems are subject of an extensive theory (see [5], problem F14, and references therein). It is interesting, therefore, that in a special case, integral identities with monomials that we considered correspond to exact covering systems.

Consider integral identities of type (4.11) with $\sum n_i = 2$, $n_k = n > 0$, and $n_i < 0$ for $i = 1, ..., k - 1$. If $F = \mathbb{R}$, we assume that n is odd (this is clearly necessary for

existence of integral identities, since the multiplication formula over $\mathbb{R}$ is valid only for multiplication by an odd number).

We will realize $\mathbb{Z}/n\mathbb{Z}$ as the set of integers $\{0, ..., n-1\}$ and parametrize covering systems by sequences $(p_1, ..., p_{k+1})$ where p_i is the biggest element of the i-th coset of the covering.

Theorem 5.3. *Integral identities of type (4.11) with such data for $F = \mathbb{C}$ are (up to rescaling a and b) in 1-1 correspondence with exact covering systems of type $(-n_1, ..., -n_{k-1}, 1, 1)$. Moreover, the formula corresponding to the covering system $(p_1, ..., p_{k+1})$ has the following parameters:*

$$\lambda_k = \mathrm{Nm}^{n-1-p_k}, \ \lambda_j = \nu_F^{-1} \, \mathrm{Nm}^{\frac{(p_j - p_k)n_j}{n}}, \tag{5.9}$$

$$\eta_k = \mathrm{Nm}^{p_{k+1}}, \eta_j = \nu_F^{-1} \, \mathrm{Nm}^{\frac{(p_{k+1} - p_j)n_j}{n}}. \tag{5.10}$$

Proof. It is clear from the divisorial relation of the Theorem 5.2 that to any integral identity with the given data, there corresponds a canonical exact covering system. Indeed, consider the divisor $D_{\mu_k^{-1}\nu_F, n}$. According to Theorem 5.2, it is exactly covered by the divisors $D_{\mu_i, -n_i}$, $i = 1, .., k-1$, $D_{1,1}$, and $D_{\chi^{-1}, 1}$. Let us regard the divisor $D_{\mu_k^{-1}\nu_F, n}$ as $\mathbb{Z}/n\mathbb{Z}$ by declaring $\mu_k^{-1}\nu_F$ to be the unit. Then the other divisors define an exact covering of the group.

Conversely, given an exact covering system, the characters λ_i, $i < k$, are uniquely determined by λ_k from the condition of cancelation of zeros and poles. Moreover, λ_k itself is uniquely determined, because the Gamma-factor corresponding to the divisor $D_{1,1}$ does not have a shift by a character. In fact, a direct computation shows that λ_i are given by the explicit formulas in the theorem. The same computation shows that $\gamma = \nu_F \, \mathrm{Nm}^{(p_{k+1} - p_k)/n}$, which allows to compute η_i from the formula $\lambda_i \eta_i \nu_F = \gamma^{n_i}$. The theorem is proved. $\square$

Theorem 5.4. *Integral identities corresponding to exact covering systems hold not only in the weak, but also in the strong sense.*

Proof. It is clear from Theorem 5.3 that λ_k is a monomial, so it is always strongly regular. Therefore, by Theorem 4.9, it suffices to check that there is no $j \leq k-1$ and singular characters r_j, r_k such that $(\lambda_j r_j^{-1})^n = (\lambda_k r_k^{-1})^{n_j}$. Suppose such j, r_j, r_k exist. Using the explicit formulas for λ_i, it is easy to deduce from this that $2 + p_j \geq n(1 - 1/n_j)$, which is impossible. The theorem is proved. $\square$

5.5. **An example.** Consider the simplest example of the above theory: $k = 2$, $n = n_2 = 3$, $n_1 = -1$, $F = \mathbb{R}$. Applying the theorems we have proved, we obtain the following result.

Theorem 5.5. *Consider the distribution*

$$\psi(ix^3/y)\lambda_2(x)\lambda_1(y). \tag{5.11}$$

(i) Such a distribution has an elementary Fourier transform in the weak sense if an only if it is one of the following six distributions:

$G_1 = e^{ix^3/y}|y|^{-1/3}; G_2 = e^{ix^3/y}|y|^{-2/3} sign(y); G_3 = e^{ix^3/y}x|y|^{-4/3} sign(y);$

$G_4 = e^{ix^3/y}x^2|y|^{-5/3}; G_5 = e^{ix^3/y}x|y|^{-2/3} sign(y); G_6 = e^{ix^3/y}x^2|y|^{-4/3} sign(y);$

(ii) The Fourier transform acts on these distributions by

$$\hat{G}_j(x,y) = \pm iG_{s(j)}(x,-y/27),$$

where the sign is $+$ for $j = 2,3,5,6$, and $-$ for $1,4$. and s is the involution of $\{1,2,3,4,5,6\}$ given by $s = (13)(45)$. These identities hold in the strong sense.

5.6. Calculation of Gamma functions of prehomogeneous vector spaces over $\mathbb{R}$ and $\mathbb{C}$.

Now we want to study integral identities for prehomogeneous vector spaces. For this purpose, we need an explicit expression for Gamma functions of these spaces. Luckily, if $F = \mathbb{R}$ or $\mathbb{C}$ then the Gamma function (up to a constant) can be found in a number of cases using Bernstein's polynomial. This is well known (see e.g. [12]), but we will give the argument for the reader's convenience.

For simplicity we restrict ourselves to the case $F = \mathbb{C}$.

Consider the (multivalued) function $f_*(\partial)f^{s+1}$ for $s \in \mathbb{C}$. It is easy to see that there exists a unique monic polynomial $\mathbf{b}(s)$ of degree D such that $f_*(\partial)f^{s+1} = \mathbf{b}(s)f^s$ (Bernstein's polynomial). In the case of prehomogeneous vector spaces that we are considering, Bernstein's polynomial was introduced by Sato in the sixties and is called "Sato's b-function" ([11]). For irreducible regular spaces, the b-functions were computed by Kimura in [9].

One has (for generic s):

$$f_*(\partial)\lambda_{s+2,n}(f) = \mathbf{b}(\frac{s-n}{2})\lambda_{s+1,n-1}(f), \overline{f_*(\partial)}\lambda_{s+2,n}(f) = \mathbf{b}(\frac{s+n}{2})\lambda_{s+1,n+1}(f). \tag{5.12}$$

Now let us find the Gamma function. We have

$$\widehat{\overline{f_*(\partial)}h}(x) = 2^{-D}(-i)^D f_*(x)\hat{h}(x), \widehat{f_*(\partial)h}(x) = 2^{-D}(-i)^D \overline{f_*(x)}\hat{h}(x).$$

Therefore, we have

$$\mathbf{b}(\frac{s \mp n}{2} - \frac{M}{D})\Gamma^V(\lambda_{s-1,n\pm 1}) = 2^{-D}(-i)^D\Gamma^V(\lambda_{s,n}).$$

Remark. The last equation and the functional equations for the Gamma functions imply that

$$\mathbf{b}(s - M/D) = (-1)^D\mathbf{b}(1 - s).$$

This shows that the collection of roots of the polynomial $\mathbf{b}(s)$ is symmetric with respect to the point $-\frac{1}{2}(1 + \frac{M}{D})$.

From now on we will assume that M/D is an integer, and that the roots of $\mathbf{b}(s)$ are integers or half integers. This condition is satisfied for a number of cases in [9]. In this case, the obtained difference equation allows one to deduce a formula for the Gamma function (up to a scalar). Namely, we have

Proposition 5.6.

$$\Gamma^V(\lambda_{s,n}) = C_V \prod_{j=1}^{D} \Gamma^F(\lambda_{s+2(s_j-M/D),n}),$$

where $\mathbf{b}(s) = \prod_i (s + s_i)$, and C_V is a constant.

Proof. Denote the proportionality coefficient between the LHS and the RHS of (5.6) by $C_V(s,n)$. It is obvious that this function is periodic: $C_V(s+1, n\pm 1) = C_V(s,n)$ and satisfies $C_V(n,s)C_V(-n,\frac{2M}{D} - s) = 1$ (by virtue of the functional equation). Thus, it suffices to show that $C_V(s,0)$ is constant.

Since $\Gamma^V(\lambda_{s-2,0}) = c \cdot \mathbf{b}(1-s)^{-2}\Gamma^V(\lambda_{s,0})$, and $\Gamma^V(\lambda_{s,0})$ is holomorphic for large positive $\mathrm{Re}(s)$, the poles of $\Gamma^V(\lambda_{s,0})$ can only arise at points of the form $2(s_j + r)$, where r is an integer. Since s_j are integers or half integers, this means that all the poles are integers. Similarly, because $1/\Gamma^V$ is holomorphic for large negative $\mathrm{Re}(s)$, we find that all the zeros are integers. The same clearly holds for the prduct of the Gamma functions on the right hand side of (5.6). Therefore, the same holds for $C_V(s,0)$.

Thus, we have: the meromorphic function $h(s) := C_V(s,0)$ is periodic with period 2, its zeros and poles are integers, and $C_V(s,0)C_V(-s,0) = 1$ (since $2M/D$ is an even integer). This implies that the function must have no zeros and no poles, since any zero of this function will also be its pole, and vice versa.

Given this, an asymtotic analysis for large s (using the stationary phase approximation) shows that $C_V(n,s)$ is a constant. Another way to show it (knowing that C_V has no zeros or poles) is to use the result of [12], which shows that C_V is a trigonometric function. $\square$

5.7. Identities for prehomogeneous vector spaces.
Let W be a prehomogeneous vector space of dimension M over $F = \mathbb{C}$, satisfying the conditions that we imposed in the previous section. Let f be its relative invariant, of degree D. Consider the space $V = V_1 \oplus ... \oplus V_{D-1}$, where $V_{D-1} = W$ and $V_j = F$ for $j = 1, ..., D-2$. We will denote elements of W by x and elements of V_j by t_j for $j = 1, ..., D-2$. Let $f_j = t_j$, $j = 1, ..., D-2$, and $f_{D-1} = f$. Let $n_1 = ... = n_{D-2} = -1$ and $n_{D-1} = D$. We are interested in identities of type (4.19) arising in this situation.

Theorem 5.7. *Identities of type (4.19) for the described data, up to rescaling a and b, correspond to functions $\sigma : \{1, ..., D\} \to \mathbb{Q}$ such that $\prod_j (s + \sigma(j))$ equals the b-function $\mathbf{b}(s)$ of the space W. More precisely, for any such function σ the parameters λ_j, η_j of the corresponding identity (4.19) are given by the formula*

$$\lambda_i = \nu_F^{\sigma(i) - \sigma(D-1) - 1}, i \leq D-2, \ \lambda_{D-1} = \nu_F^{\sigma(D-1) - M/D}, \qquad (5.13)$$

$$\eta_i = \nu_F^{\sigma(D) - \sigma(j) - 1}, i \leq D-2, \ \eta_{D-1} = \nu_F^{1-\sigma(D)} \qquad (5.14)$$

Proof. The proof is analogous to the proof for usual monomials. $\square$

5.8. Example.
Let W be the 27-dimensional prehomogeneous vector space over $\mathbb{C}$ from example 6 in Section 3.4. Let W^* be the dual space, and f_* the multiplicative Legendre transform of f. The b-function of this space equals $(s+1)(s+5)(s+9)$ (see [9]). Therefore, from the previous theorem we obtain the following result

Theorem 5.8. *Consider the distribution $\psi(iRef(x)/y)|f(x)|^p|y|^q$ on $W \oplus \mathbb{C}$ ($p, q \in \mathbb{C}$).*

(i) Such a distribution has an elementary Fourier transform in the weak sense if and only if (q, p) takes one of the following six values: (-10,-8); (-18,0); (6,-16); (-10,0); (6,-8); (14,-16).

(ii) The Fourier transform in the weak sense maps any of these distributions to a distribution of the same type on $W^* \oplus \mathbb{C}$ (up to a scalar). The action of the Fourier transform on the parameters (q, p) is given by the following involution:

$(-10, -8) \to (14, -16)$, $(-18, 0) \to (6, -8)$, $(6, -16) \to (6, -16)$, $(-10, 0) \to (-10, 0)$.

The identity corresponding to $(-10, 0) \to (-10, 0)$ holds in the strong sense.

Remark. The fact that the identity corresponding to $(-10,0) \to (-10,0)$ holds in the strong sense is proved similarly to the case of usual monomials.

Remark. Note that the integral identities other than $(-10,0) \to (-10,0)$ cannot be understood in the strong sense, because the distributions we considered have poles on the divisor $f(x) = 0$ and therefore are not well-defined without a regularization. So to replace the identities we considered by identities in the strong sense one would first need to regularize one or both sides. We leave this beyond the scope of this paper.

6. IDENTITIES OVER LOCAL NON-ARCHIMEDEAN FIELDS

6.1. **Local constants and Gamma functions.** Let F be a local non-archimedean field. Let $\mathcal{O} \subset F$ be the valuation ring, $\pi \in F^*$ be a uniformizing. A character $\chi \in X(F^*)$ is called unramified if $\chi|_{\mathcal{O}^*} = 1$. The local L-factors are defined as the following meromorphic functions of $\chi \in X(F^*)$:

$$L(\chi) = \begin{cases} 1 & \text{if } \chi \text{ is ramified,} \\ (1 - \chi(\pi))^{-1} & \text{if } \chi \text{ is unramified.} \end{cases}$$

Note that $L(\chi)$ has a unique pole at the trivial character. Then we have

$$\Gamma^F(\chi) = \frac{L(\chi)}{L(\chi^{-1}\nu_F)} \epsilon(\chi^{-1}\nu_F)$$

where $\epsilon(\chi) = \epsilon(\chi, \psi) \in \mathbb{C}^*$ are the local constants considered by Deligne in [3]. With our choice of the Haar measure dx on F we have $\epsilon(\chi) = 1$ if χ is unramified. If χ is ramified then

$$\epsilon(\chi, \psi) = \int_{F^*} \chi^{-1}(x)\psi(x)dx := \sum_n \int_{v(x)=n} \chi^{-1}(x)\psi(x)dx.$$

Note that the functional equation (4.3) for Gamma function implies that

$$\epsilon(\chi)\epsilon(\chi^{-1}\nu_F) = \chi(-1).$$

6.2. **Identities between local constants.** Recall that for a local field F the Weil group W_F is defined as the preimage of the subgroup generated by the Frobenius under the surjective homomorphism $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{k}/k)$ where k is the residue field of F. The local class field theory provides an isomorphism between the abelianization of the Weil group W_F and F^*. Following [3] we normalize this isomorphism in such a way that uniformizings in F^* correspond to liftings of the inverse of the Frobenius. Note that for every finite separable extension $F \subset E$ we have a commutative diagram

$$\begin{array}{ccc} W_E & \longrightarrow & E^* \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{Nm}_{E/F}} \\ W_F & \longrightarrow & F^* \end{array} \qquad (6.1)$$

where i is the natural inclusion. Thus, to a character χ of F^* we can associate a one-dimensional representation of W_F. We denote by $[\chi]$ the corresponding element of the representation ring $R(W_F)$ (of finite-dimensional continuous complex

representations of W_F). If λ is a character of F^* and $F \subset E$ is a finite separable extension then $[\lambda \circ \mathrm{Nm}_{E/F}] = \mathrm{Res}[\lambda]$ where $\mathrm{Res} : R(W_F) \to R(W_E)$ is the restriction homomorphism.

The principal theorem of [3] (Theorem 4.1) says that the map $[\chi] \mapsto \epsilon(\chi)$ extends to a homomorphism $V \mapsto \epsilon(V)$ from $R(W_F)$ to $\mathbb{C}^*$, such that for a finite separable extension E of F and for any virtual representation V of W_E of dimension 0 one has

$$\epsilon(\mathrm{Ind}_{W_E}^{W_F} V, \psi) = \epsilon(V, \psi \circ \mathrm{Tr}_{E/F})$$

(where both sides do not depend on a choice of Haar measures). If the dimension of V is not zero then we can write

$$\epsilon(\mathrm{Ind}_{W_E}^{W_F} V) = \lambda(E/F)^{\dim V} \cdot \epsilon(V)$$

where $\lambda(E/F) = \epsilon(\mathrm{Ind}_{W_E}^{W_F}[\nu_F^s])$ (which doesn't depend on s).

On the other hand, one can extend local L-factors $L(\chi)$ to a homomorphism from $R(W_F)$ to the group of non-zero meromorphic functions on $X(F^*)$ by setting

$$L(V, \lambda) = \det(1 - \mathrm{Frob}^{-1}|_{(\lambda \otimes V)^I})^{-1}$$

where $I \subset W_F$ is the inertia subgroup (see [3], 3.5). Moreover, for a finite separable extension $F \subset E$ and $[V] \in R(W_E)$ one has

$$L(\mathrm{Ind}_{W_E}^{W_F}[V], \lambda) = L([V], \lambda \circ \mathrm{Nm}_{E/F})$$

(see [3], Prop. 3.8).

Proposition 6.1. *Let F_i be finite separable extensions of a local field F and for every i let χ_i be a character of F_i^*. Assume that we have a linear relation*

$$\sum_i n_i \mathrm{Ind}[\chi_i] = 0$$

between the induced representations $\mathrm{Ind}[\chi_i]$ of W_F, where $n_i \in \mathbb{Z}$. Then one has the following identity:

$$\prod_i (\lambda(F_i/F) \Gamma^{F_i}(\chi_i(\lambda \circ \mathrm{Nm}_{F_i/F}))^{n_i}) = 1$$

for $\lambda \in X(F^*)$.

Proof. It suffices to notice that we also have the linear relation

$$\sum_i n_i \mathrm{Ind}[\chi_i^{-1}\nu_{F_i}] = 0$$

(since $\nu_{F_i} = \nu_F \circ \mathrm{Nm}_{F_i/F}$) and apply the above properties of L-factors and ϵ-constants. $\square$

Here is an example of an identity provided by the above proposition. Let $F \subset E$ be a finite cyclic extension. Then for any character λ of F^* we have the relation

$$\mathrm{Ind}_{W_E}^{W_F}[\lambda \circ \mathrm{Nm}_{E/F}] = \sum_{\chi \circ \mathrm{Nm}_{E/F} = 1} [\chi \cdot \lambda]$$

where the sum is taken over all characters χ of F^* which are trivial on $\mathrm{Nm}_{E/F}(E^*)$ (note that this subgroup has index $[E : F]$ in F^*). Hence we derive the identity

$$\lambda(E/F) \cdot \Gamma^E(\lambda \circ \mathrm{Nm}_{E/F}) = \prod_{\chi \circ \mathrm{Nm}_{E/F} = 1} \Gamma^F(\chi \cdot \lambda).$$

Remark. In the simplest case $[E:F]=2$ this formula appears in [4].

More generally, let $F \subset E$ be a finite cyclic extension. For every $n > 0$ such that $n|[E:F]$ let $F_n \subset E$ be the cyclic subextension of degree n over F. Let us denote by $X_{F,E}$ the subgroup in $X(F^*)$ consisting of characters of the form $\nu_F^m \chi$ where $m \in \mathbb{Z}$, $\chi \in X(F^*)$ is trivial on $\mathrm{Nm}_{E/F}(E^*) \subset F^*$. Let $Div(X_{F,E})$ be the group of divisors on $X_{F,E}$. For every character χ in $X_{F_n,E}$ let us define the divisor $D(\chi) \in Div(X_{F,E})$ as follows:

$$D(\chi) = \sum_{\lambda : \lambda \circ \mathrm{Nm}_{F_n/F} = \chi} (\lambda).$$

Note that the homomorphism

$$X_{F,E} \to X_{F_n,E} : \lambda \mapsto \lambda \circ \mathrm{Nm}_{F_n/F}$$

is surjective, hence the divisor $D(\chi)$ has degree n. As an additive character on F_n let us take $\psi_n = \psi \circ \mathrm{Tr}_{F_n/F}$. Then extending the Gamma function to divisors multiplicatively we can write

$$\lambda(F_n/F) \cdot \Gamma^{F_n}(\chi(\lambda \circ \mathrm{Nm}_{F_n/F})) = \Gamma^F(\lambda D(\chi)).$$

for $\chi \in X_{F_n,E}$, $\lambda \in X(F^*)$, where for every divisor D we denote by λD the divisor obtained from D by applying the shift $\mu \mapsto \lambda \mu$.

Now let $(d_1, \dots, d_k)$ be a sequence of positive numbers dividing $[E:F]$, $(m_1, \dots, m_k)$ be a sequence of integers. For every $i = 1, \dots, k$ let χ_i be a character in $X_{F_{d_i},E}$. Assume that

$$\sum_i m_i D(\chi_i) = 0.$$

Then for every $\lambda \in X(F^*)$ we have

$$\prod_i \lambda(F_{d_i}/F)^{m_i} \Gamma^F(\chi_i(\lambda \circ \mathrm{Nm}_{F_{d_i}/F}))^{m_i} = 1. \tag{6.2}$$

Using the functional equation for Gamma functions we can rewrite this result slightly differently. For $e = \pm 1$ and a character $\chi \in X_{F_n,E}$ let us set

$$D(\chi, e) = \begin{cases} D(\chi), & \text{if } e = 1, \\ -D(\chi^{-1}\nu_{F_n}), & \text{if } e = -1. \end{cases}$$

Then for a sequence $(e_1, \dots, e_k)$ where $e_i = \pm 1$ the relation

$$\sum_i D(\chi_i, e_i) = 0$$

implies the identity

$$\prod_i \Gamma^F(\chi_i(\lambda^{e_i} \circ \mathrm{Nm}_{F_{d_i}/F})) = C \cdot \lambda(\prod_i e_i)$$

for $\lambda \in X(F^*)$, where the constant $C \in \mathbb{C}^*$ doesn't depend on λ.

6.3. **Identities for cyclic extensions.** Let $F \subset E$ be a cyclic extension of local fields, $F_i \subset E$ be a subextension of degree d_i over F $(i = 1, \dots, k)$. For every $i = 1, \dots, k$ let χ_i be a character in $X_{F_i,E}$, e_i be either 1 or -1. Below we denote by 1_F the trivial character of F^*. The following theorem follows easily from the above identities for Gamma functions and from theorem 4.7.

Theorem 6.2. *In the above situation assume that one of the following identities in $Div(X_{F,E})$ holds:*

   1. $D(1_F, 1) + \sum_i D(\chi_i, -e_i) = D(\xi, -1)$,
   2. $D(1_F, 1) + \sum_i D(\chi_i, -e_i) = D(\xi, 1)$,

for some $\xi \in X_{F,E}$. Then we have the following identity between distributions on $\oplus_i F_i$ in the weak sense:

$$\mathcal{F}(\prod_i (\chi_i \nu_{F_i}^{-1})(x_i)\psi(a\prod_i \mathrm{Nm}_{F_i/F}(x_i)^{e_i})) = C \cdot \prod_i \eta_i(x_i)\psi(b\prod_i \mathrm{Nm}_{F_i/F}(x_i)^{ee_i})$$

for some constants $C \in \mathbb{C}^*$, $a, b \in F^*$, where $e = 1$ in the case 1, $e = -1$ in the case 2,

$$\eta_i = \chi_i^{-1}(\xi^{ee_i} \circ \mathrm{Nm}_{F_i/F}).$$

For example, the identity (1.2) corresponds to the following equality of divisors:

$$D(1_F, 1) + D(\mathcal{E}, 1) + D(\nu_E, -1) = D(\mathcal{E}\nu_F, -1).$$

Thus, we see that identity (1.2) holds in the weak sense.

Let us prove now that identity (1.2) in fact holds in the strong sense. We will give a sketch of the argument. First of all, the function $\phi_{\mathcal{E}}$ makes sense as a distribution (i.e. does not need regularization). Since 1.2 holds in the weak sense, we have $\widehat{\phi_{\mathcal{E}}} = \epsilon \phi_{\mathcal{E}} + \eta$, where η is a distribution which is a sum of a distribution concentrated on the coordinate cross and a distribution whose Fourier transform is concentrated on the cross, and ϵ is a sign. The distribution η has the same homogeneity properties as $\phi_{\mathcal{E}}$, and satisfies $\hat{\eta} = -\epsilon\eta$. From this it is easy to deduce that $\eta = c(\delta(t) - \epsilon\delta(x))$. Thus, it remains to check that $c = 0$, which can be checked by a direct calculation; for instance, one can check directly that $\hat{\phi}_{\mathcal{E}}$ is locally constant at $x = 0, t \neq 0$. Thus, (1.2) holds in the strong sense.

6.4. **One more identity.** There are more complicated examples of identities between local constants which involve non-abelian extensions (see [3], section 1). Here is an example. Let l be a positive integer, E be a Galois extension of F with Galois group G which is a central extension of $(\mathbb{Z}/l\mathbb{Z})^2$ by an abelian group Z. Let $H_1 \subset G$ (resp. H_2) be the preimages of the first (resp. the second) factor $\mathbb{Z}/l\mathbb{Z}$ in $(\mathbb{Z}/l\mathbb{Z})^2$. Let χ_i be a character of H_i for $i = 1, 2$, such that $\chi_1|_Z = \chi_2|_Z$ and this character is non-trivial on $[G, G] \subset Z$. Let $F_i \subset E$ be the subextension of F corresponding to the subgroup H_i. Then for $i = 1, 2$ we can consider χ_i as a character in $X_{F_i,E}$ and for every character $\lambda \in X(F^*)$ one has

$$\mathrm{Ind}_{W_{F_1}}^{W_F} \chi_1(\lambda \circ \mathrm{Nm}_{F_1/F}) = \mathrm{Ind}_{W_{F_2}}^{W_F} \chi_2(\lambda \circ \mathrm{Nm}_{F_2/F})$$

Hence,

$$\lambda(F_1/F)\Gamma^{F_1}(\chi_1(\lambda \circ \mathrm{Nm}_{F_1/F})) = \lambda(F_2/F)\Gamma^{F_2}(\chi_2(\lambda \circ \mathrm{Nm}_{F_2/F})) \qquad (6.3)$$

for $\lambda \in X(F^*)$. Now the following theorem follows easily from theorem 4.7.

Theorem 6.3. *In the above situation we have the following identity of distributions on $F_1 \times F_2$ in the weak sense:*

$$\mathcal{F}((\chi_1\nu_{F_1}^{-1})(x_1)\chi_2^{-1}(x_2)\psi(\frac{\mathrm{Nm}_{F_1/F}(x_1)}{\mathrm{Nm}_{F_2/F}(x_2)})) = C\cdot(\chi_2\nu_{F_2}^{-1})(x_2)\chi_1^{-1}(1_2)\psi((-1)^l\frac{\mathrm{Nm}_{F_2/F}(x_2)}{\mathrm{Nm}_{F_1/F}(x_1)}).$$

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